Subinvariant metric functionals for nonexpansive mappings

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Abstract

We investigate the existence of subinvariant metric functionals for commuting families of nonexpansive mappings in noncompact subsets of Banach spaces. Our findings underscore the practicality of metric functionals when searching for fixed points of nonexpansive mappings. To demonstrate this, we additionally investigate subsets of Banach spaces that have only nontrivial metric functionals. We particularly show that in certain cases every metric functional has a unique minimizer; thus, subinvariance implies the existence of a fixed point.

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1. Introduction

The purpose of this note is to demonstrate the utility of metric functionals in the study of fixed points of nonexpansive mappings.

A mapping $T : E \to E$ defined on a metric space (E, d) is called nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in E$. Nonexpansive mappings

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are not simply abstract objects studied by mathematicians; they also form the backbone of many important methods used by applied science practitioners. For example, nonexpansive mappings are naturally found in the design of iterative methods for optimization algorithms, nonlinear evolution equations, and control systems (see, for example, [1] and [2]). Such methods are designed as follows: one associates the original problem with a family of nonexpansive mappings $T_\mu : E \to E$ so that a common fixed point $z = T_\mu z$ (for all μ) gives a solution to the aforesaid problem, then one builds in the metric space E a sequence of points (x_μ) that under certain conditions converges to z. It is worth mentioning that the existence of a common fixed point for a given family of nonexpansive mappings is not guaranteed in general.

The main key in our research approach is the notion of a metric functional. The idea is simple and of common form within mathematics: one considers the metric space been embedded into a larger space (the set of metric functionals) where a solution to a “weaker" problem may be available, then sometimes one can show that this procedure yields a solution in the original space and formulation. This idea is rigorously defined below.

Definition 1.1. *Let (E, d) be a metric space. We denote by $\mathbb{R}^E$ the space of all functionals from E to $\mathbb{R}$ and equip it with the topology of pointwise convergence. We fix a point $o \in E$ and consider the mapping $w \mapsto \mathbf{h}_w$ from E to $\mathbb{R}^E$ defined by the formula*

$$\mathbf{h}_w(x) = d(x, w) - d(o, w) \quad \textit{for all} \quad x \in E. \tag{1.1}$$

We denote by $E^\diamondsuit$ the closure of the set $\{\mathbf{h}_w \mid w \in E\}$ in $\mathbb{R}^E$ and we call each element of $E^\diamondsuit$ a metric functional*.*

Proposition 1.2. ([3, Chapter 3]) *The following properties hold:*

1. *The mapping $w \mapsto \mathbf{h}_w$ from E to $\mathbb{R}^E$ is injective and continuous.*
2. *The space $E^\diamondsuit$ is compact and Hausdorff.*

Since each point $w \in E$ is uniquely identified with the metric functional $\mathbf{h}_w \in E^\diamondsuit$, we may view E as been embedded into the compact space $E^\diamondsuit$. For that reason, we call the metric functionals of the form (1.1) *internal*.

When a subset X of E is considered equipped with the same metric d, we can similarly build the compact space $X^\diamondsuit$. Clearly, each internal metric functional $\mathbf{h}_w \in X^\diamondsuit$ can be defined on the whole space E; thus, for each $\mathbf{h} \in X^\diamondsuit$ there exists $\widetilde{\mathbf{h}} \in E^\diamondsuit$ such that $\widetilde{\mathbf{h}}(x) = \mathbf{h}(x)$ for all $x \in X$ (see [4, Proposition 1]). In this context, *we will always assume that* $X^\diamondsuit$ is a subset of $E^\diamondsuit$ and hence operates in all points of E.

Definition 1.3. *We say that a functional* $f \in \mathbb{R}^E$ *is* subinvariant *for a mapping* $T : E \to E$ *if* $f(Tx) \leq f(x)$ *for all* $x \in E$.

It follows readily from the previous definitions that the existence of internal metric functionals which are subinvariant and the existence of fixed points coincide for nonexpansive mappings. We record here a precise statement.

Proposition 1.4. *Given a mapping* $T : E \to E$*, the following holds:*

1. *If an internal metric functional* $\mathbf{h}_z \in E^\diamondsuit$ *is subinvariant for* T*, then* $z \in E$ *is a fixed point of* T.
2. *If* T *is nonexpansive and has a fixed point* $z \in E$*, then the internal metric functional* $\mathbf{h}_z \in E^\diamondsuit$ *is subinvariant for* T.

We may face different situations with subinvariant metric functionals that are not internal. To begin with, some spaces have the metric functional that vanishes everywhere, and hence it is subinvariant for all mappings. Secondly, there may be subinvariant metric functionals that are not internal but provide the existence of fixed points. And finally, if for some metric functional we would have $\mathbf{h}(Tx) < \mathbf{h}(x)$ for all $x \in E$, then T cannot have a fixed point in E.

2. Problem statement

In what follows we will consider nonexpansive mappings that are defined on Banach spaces or their subsets equipped with the same norm. To be precise, we want to study the following problem.

> **Problem.** *Let* X *be a subset of a Banach space* E *and let* $\mathcal{F}$ *be a commuting family of nonexpansive mappings from* X *to itself. Find a metric functional* $\mathbf{h} \in X^\diamondsuit$ *that is subinvariant for all* $T \in \mathcal{F}$.

We present next a simple example that fits our problem statement and is related to the iterative methods mentioned in the previous section.

Example 2.1. Let us assume that $A : E \to E$ is a linear operator and b is some point in E. For each real number μ let us consider the mapping $T_\mu : E \to E$ defined by the formula

$$T_\mu x = (1 - \mu)x + \mu(Ax + b).$$

We notice that for all $\mu, \nu \in \mathbb{R}$ and for all $x \in E$ we have $T_\mu T_\nu x = T_\nu T_\mu x$. Now, if we assume that $\|Ax\| \leq \|x\|$ for all $x \in E$, then $\mathcal{F} = \{T_\mu \mid 0 < \mu < 1\}$ is a commuting family of nonexpansive mappings.

We can extend the previous example as follows: let $\{q_s\}$ be a family of nonvanishing polynomials and set

$$p_s(\lambda) = 1 - (1 - \lambda)q_s(\lambda).$$

Consider the affine mappings $T_s x = p_s(A)x + q_s(A)b$, where A is a bounded linear operator (see [5, Proposition 1.4.2]). Then, $\mathcal{F} = \{T_s\}$ is a commuting family. Depending on the spectrum $\sigma(A)$, it is possible that $\|p_s(A)\| \leq 1$. Collecting all such pairs q_s, p_s we have a commuting family of affine nonexpansive mappings.

Before stating our main results, we want to present another example that motivates our study.

Example 2.2. Let us assume that E is any of the Banach spaces c_0, ℓ_1, ℓ_2, ℓ_∞ consisting of real sequences $x = (x_k)_{k \geq 1}$. Let us consider the mapping $T : E \to E$ given by the formula

$$T(x_1, x_2, x_3, \dots) = (1, x_1, x_2, x_3, \dots).$$

It is clear that T is both affine and isometric. From purely algebraic considerations, we observe that the only sequence which is mapped back to itself is $z = (1, 1, 1, \dots)$. Hence, T has a unique fixed point in the space $E = \ell_\infty$ and $\mathbf{h}_z$ is a unique subinvariant metric functional. Now, z is not in any of the smaller spaces, but we can construct subinvariant metric functionals for T by considering the "truncated" vectors $a_n = T^n 0$ and taking limits of $\mathbf{h}_{a_n}$. We will show in Proposition 4.3 that if $E = \ell_2$ then the *only* metric functional that is subinvariant for T vanishes identically. Since there is no other subinvariant metric functional, T has no fixed points in ℓ_2. We will show also in Proposition 4.3 that if $E = \ell_1$ then there are infinitely many metric functionals that are subinvariant for T, all of them nontrivial. For example, the unbounded sequence (a_n) in ℓ_1 generates the metric functional

$$\mathbf{h}(x) = \sum_{k=1}^{\infty}(|x_k - 1| - 1),$$

which is subinvariant for T. In fact, we have $\mathbf{h}(Tx) = \mathbf{h}(x) - 1$ for all $x \in \ell_1$, and hence T has no fixed points in ℓ_1. Finally, if $E = c_0$ then the sequence (a_n) in c_0 generates the metric functional

$$\mathbf{h}(x) = \sup_{k \geq 1}|x_k - 1| - 1,$$

which is subinvariant for T.

Remark 2.3. A theorem of Gaubert and Vigeral [6] implies the existence of a metric functional that is subinvariant for each element of the family $\mathcal{F} = \{T^n \mid n \geq 1\}$, where $T : X \to X$ is nonexpansive and X is a star-shaped subset of E. Karlsson showed a similar result in [7, Proposition 13]. Their approaches, however, do not seem to be adaptable for general commuting families of nonexpansive mappings.

3. Main results

Our main results are stated here and their proofs are given in Section 5. For a mapping $T : X \to X$ we set $m(T, X) = \inf\{\|x - Tx\| \mid x \in X\}$.

Theorem 3.1. *Let E be a Banach space and let $\mathcal{F}$ be a commuting family of affine nonexpansive mappings from E to itself. If X is a nonempty convex subset of E with the property that for all $T \in \mathcal{F}$ we have $TX \subset X$ and $m(T, X) = 0$, then there exists a metric functional $\mathbf{h} \in X^\diamond$ such that for all $x \in E$ and for all $T \in \mathcal{F}$ we have*

$$\mathbf{h}(Tx) \leq \mathbf{h}(x).$$

In other words, $\mathbf{h}$ is a subinvariant metric functional for all $T \in \mathcal{F}$.

Remark 3.2. By a theorem of Kohlberg and Neyman [8], we know that if $T : X \to X$ is a nonexpansive mapping defined on a convex subset X of a Banach space, then for all $y \in X$ we have $m(T, X) = \lim_{n\to\infty}\|T^n y\|/n$. Thus, the assumption $m(T, X) = 0$ for all $T \in \mathcal{F}$ in Theorem 3.1 holds whenever there is a vector $y \in X$ such that for all $T \in \mathcal{F}$ the aforementioned limit equals zero. This clearly holds for example when X is bounded.

The following statement holds true for nonexpansive mappings, assuming neither affinity nor convexity, with the condition that there is a vector x_0 such that

$$\lim_{n\to\infty} \|T^n x_0 - T^{n+1} x_0\| = 0. \tag{3.1}$$

Theorem 3.3. *Let E be a Banach space and let $\mathcal{F}$ be a commuting family of nonexpansive mappings from E to itself. If X is a nonempty subset of E such that $TX \subset X$ for all $T \in \mathcal{F}$ and has a vector x_0 such that* (3.1) *holds for all $T \in \mathcal{F}$, then there exists a metric functional $\mathbf{h} \in X^\diamondsuit$ that is subinvariant for all $T \in \mathcal{F}$.*

We emphasize that our results are valid without any compactness assumption as opposed to what is assumed in classical fixed-point theorems such as those obtained by Markov [9], Kakutani [10], DeMarr [11], and Browder [12]. As it was indicated in Proposition 1.4, for every nonexpansive mapping T, the existence of internal metric functionals that are subinvariant for T is equivalent to the existence of fixed points of T. Internal metric functionals appear naturally when T maps a compact set to itself.

Sometimes we may be able to conclude the existence of a fixed point even when the subinvariant metric functional is not internal. This happens for example in cases where one knows all the metric functionals on the considered space. Gutiérrez [13] showed explicit formulas for all the metric functionals on the ℓ_p spaces with $1 \le p < \infty$, we recall those formulas in Section 4. Having these available, the existence of a common fixed point follows from Theorem 3.1 as shown below.

Corollary 3.4. *Assume that $1 \le p < \infty$. Suppose that $\mathcal{F}$ is a commuting family of affine nonexpansive mappings from ℓ_p to itself. If there exists a nonempty bounded subset B of ℓ_p such that for all $T \in \mathcal{F}$ we have $TB \subset B$, then the family $\mathcal{F}$ has a common fixed point in ℓ_p. More precisely, there exists a vector $z \in \ell_p$ such that for all $T \in \mathcal{F}$ we have $Tz = z$.*

Proof. Let $E = \ell_p$ and let X be the convex hull of B. Thus, X is a nonempty bounded convex subset of E such that $TX \subset X$ for all $T \in \mathcal{F}$. It follows from Remark 3.2 that for all $T \in \mathcal{F}$ we have $m(T, X) = 0$. By Theorem 3.1, there exists a metric functional $\mathbf{h} \in X^\diamondsuit$ that is subinvariant for all $T \in \mathcal{F}$. This metric functional $\mathbf{h}$ must be of the form (4.1) when $p > 1$ (see Theorem 4.1) or internal when $p = 1$ (see Theorem 4.2). Thus, there exists a vector $z \in E$ such that for all $x \in E$ and for all $T \in \mathcal{F}$ we have $\|Tx - z\|_p \le \|x - z\|_p$. Therefore, the vector $z \in E$ is the common fixed point of the family $\mathcal{F}$. $\square$

3.1. A fixed point theorem

The fixed point z shown in Corollary 3.4 is not necessarily a point in the set B. We need additional properties if we insist on looking for a fixed point inside a given set. A key property that holds in all the ℓ_p spaces with $1 \leq p < \infty$ is *the Opial property* [14]. We recall this property in Definition 4.21.

The following statement could serve as a model result to obtain a fixed point from a subinvariant metric functional.

Theorem 3.5. *Let X be a nonempty weakly compact subset of a Banach space having the Opial property. If a mapping $T : X \to X$ has a subinvariant metric functional $\mathbf{h} \in X^\diamondsuit$, then T has a fixed point in X.*

4. Metric functionals

Metric functionals on Banach spaces were investigated, for example, in [15], [13], [16], [17], [18], [19], [20]. We recall here explicit formulas for all metric functionals on the ℓ_p spaces with $1 \leq p < \infty$, where the norm of every real sequence $x = (x_k)_{k\geq 1}$ is defined by

$$\|x\|_p = \Big(\sum_{k\geq 1} |x_k|^p\Big)^{1/p}.$$

Theorem 4.1. ([13, Section 5]) *Assume that $1 < p < \infty$. We have $\mathbf{h} \in (\ell_p)^\diamondsuit$ if and only $\mathbf{h}$ is either a continuous linear functional with norm at most 1, or a functional of the form*

$$\mathbf{h}(x) = \big(\|x - z\|_p^p + c^p - \|z\|_p^p\big)^{1/p} - c \tag{4.1}$$

for some $z \in \ell_p$ and some $c \in \mathbb{R}$ with $c \geq \|z\|_p$. Moreover, if X is a bounded subset of ℓ_p then each metric functional $\mathbf{h} \in X^\diamondsuit$ is of the form (4.1).

Theorem 4.2. ([13, Section 3]) *We have $\mathbf{h} \in (\ell_1)^\diamondsuit$ if and only if there exists a subset I of $\mathbb{N}$, an element ε of $\{-1, 1\}^I$, and an element z of $\mathbb{R}^{\mathbb{N}\setminus I}$ such that for all $x \in \ell_1$ we have*

$$\mathbf{h}(x) = \sum_{i\in I} \varepsilon_i x_i + \sum_{i\notin I} (|x_i - z_i| - |z_i|). \tag{4.2}$$

Moreover, if X is a bounded subset of ℓ_1 then each metric functional $\mathbf{h} \in X^\diamondsuit$ is of the form $\mathbf{h}(x) = \|x - w\|_1 - \|w\|_1$ for some $w \in \ell_1$.

Now we proceed to prove the claims associated with the mapping T given in Example 2.2.

Proposition 4.3. *For the mapping $T : E \to E$ defined in* Example 2.2 *the following holds:*

1. *If $E = \ell_2$ then the only subinvariant metric functional for T vanishes identically.*
2. *If $E = \ell_1$ then there are infinitely many subinvariant metric functionals for T, all of them nontrivial.*

Proof. To show the first statement we use Theorem 4.1 as follows. First, let us assume that any metric functional of the form (4.1) is subinvariant for T. This implies that there is a vector $z \in \ell_2$ such that for all $x \in \ell_2$ we have $\|Tx - z\|_2 \le \|x - z\|_2$, which is a contradiction as T has no fixed points in ℓ_2. Thus, the remaining candidates must be linear functionals $\mathbf{h}(x) = \langle x, z\rangle$, where $z \in \ell_2$ with $\|z\|_2 \le 1$. Now, $\mathbf{h}$ is subinvariant for T if and only if $\langle x - Tx, z\rangle \ge 0$ for all $x \in \ell_2$. Denote $z = (z_1, z_2, \dots)$. If $z_1 \ne 0$, let us define $t = (z_1 - 1)/z_1$. Since z is a vector in ℓ_2, there exists an integer $n > 1$ such that $|z_n| \le 1/(2|t| + 1)$. Let us define a vector $x = (x_1, x_2, \dots)$ in ℓ_2 by $x_j = t$ for $j < n$ and $x_j = 0$ for $j \ge n$. Then, we have $\langle x - Tx, z\rangle \le -1/2$, and hence the metric functional $\mathbf{h}$ is not subinvariant. Finally, if $m > 1$ is the smallest index j such that $z_j \ne 0$, we consider the vector $x = (x_1, x_2, \dots)$ in ℓ_2 such that $x_{m-1} = 1/z_m$ and $x_j = 0$ for $j \ne m - 1$. Then, we have $\langle x - Tx, z\rangle = -1$.

To show the second statement we use Theorem 4.2. We notice that the $0-$functional is not a metric functional on ℓ_1 (see (4.2)). Now, for each integer $N \ge 1$ let us consider the metric functional $\mathbf{h}^{(N)} \in (\ell_1)^{\diamondsuit}$ given by the formula

$$\mathbf{h}^{(N)}(x) = \sum_{j=N+1}^{\infty} (-x_j) + \sum_{j=1}^{N} \big(|x_j - 1| - 1\big).$$

Then, for all $x \in \ell_1$ we have $\mathbf{h}^{(N)}(x) - \mathbf{h}^{(N)}(Tx) = |x_N - 1| + x_N \ge 1$. $\square$

Notice that if T is nonexpansive and has a unique fixed point z, then the internal metric functional $\mathbf{h}_z$ need not be the only subinvariant functional for T. In fact, if S is the forward shift in ℓ_2, then with all $c \ge 0$ the metric functional $\mathbf{h}(x) = (\|x\|_2^2 + c^2)^{1/2} - c$ (see (4.1)) is invariant for S.

4.1. Properties of metric functionals

As we observed previously, there are metric spaces which have the metric functional that vanishes identically. It is therefore reasonable to determine conditions under which a metric space has only nontrivial metric functionals. To this end, we introduce here some relevant concepts.

Definition 4.4. *We say that a metric space X has* the zero-free property*, ZFP, if for each metric functional $\mathbf{h} \in X^\diamondsuit$ there exists a point $x \in X$ such that $\mathbf{h}(x) \neq 0$.*

Definition 4.5. *We say that a metric space X has* the unique minimizer property*, UMP, if for each metric functional $\mathbf{h} \in X^\diamondsuit$ there exists a point $a \in X$ such that for all $x \in X \setminus \{a\}$ we have $\mathbf{h}(a) < \mathbf{h}(x)$.*

The following implication is immediate.

Proposition 4.6. *If a metric space has UMP then it has ZFP.*

We emphasize that ZFP and UMP depend truly on the metric. To show this, let us assume that (A, d_A) and (B, d_B) are metric spaces and let X_p denote the direct sum $A \oplus B$ equipped with different metrics d_p:

$$d_p = \begin{cases} (d_A^p + d_B^p)^{1/p} & \text{for } \ 1 \leq p < \infty, \\ \max\{d_A, d_B\} & \text{for } \ p = \infty. \end{cases}$$

Proposition 4.7. *For the metric space X_1 the following properties hold:*

1. *We have $\mathbf{h} \in (X_1)^\diamondsuit$ if and only if there are two metric functionals $\mathbf{h}^A \in A^\diamondsuit$ and $\mathbf{h}^B \in B^\diamondsuit$ such that $\mathbf{h}(x) = \mathbf{h}^A(a) + \mathbf{h}^B(b)$ for all $x = (a, b) \in X_1$.*
2. *The metric space X_1 has ZFP if and only if at least one of A and B has ZFP.*
3. *The metric space X_1 has UMP if and only if both A and B have UMP.*

Proof. Let us fix two points $a_0 \in A$ and $b_0 \in B$. The point $(a_0, b_0) \in X_1$ is fixed when we build the compact space $(X_1)^\diamondsuit$.

The first statement follows from the formula

$$d_1((a, b), (w, z)) - d_1((a_0, b_0), (w, z)) = \mathbf{h}^A_w(a) + \mathbf{h}^B_z(b),$$

where $\mathbf{h}^A_w(a) = d_A(a, w) - d_A(a_0, w)$ and $\mathbf{h}^B_z(b) = d_B(b, z) - d_B(b_0, z)$.

Let us consider the second statement. If both A and B have identically vanishing metric functionals, their sum also vanishes. Now, let us assume that A has ZFP and consider a metric functional $\mathbf{h} = \mathbf{h}^A + \mathbf{h}^B$. We know that there exists $\hat{a} \in A$ such that $\mathbf{h}^A(\hat{a}) \neq 0$. Since $\mathbf{h}^B(b_0) = 0$, we have $\mathbf{h}(\hat{a}, b_0) = \mathbf{h}^A(\hat{a}) \neq 0$. If B has ZFP instead, then we have $\mathbf{h}(a_0, \hat{b}) \neq 0$ for some $\hat{b} \in B$.

To show the last statement, let us assume first that both A and B have UMP. Then, for every metric functional $\mathbf{h} = \mathbf{h}^A + \mathbf{h}^B$ there are $\hat{a} \in A$ and $\hat{b} \in B$ such that $\mathbf{h}^A(\hat{a}) < \mathbf{h}^A(a)$ for all $a \neq \hat{a}$ and $\mathbf{h}^B(\hat{b}) < \mathbf{h}^B(b)$ for all $b \neq \hat{b}$. This implies that $\mathbf{h}(\hat{a}, \hat{b}) < \mathbf{h}(a, b)$ for all $(a, b) \neq (\hat{a}, \hat{b})$. Now, let us assume that X_1 has UMP and consider two metric functionals $\mathbf{h}^A \in A^\diamondsuit$ and $\mathbf{h}^B \in B^\diamondsuit$. Since X_1 has UMP, there are two points $\hat{a} \in A$ and $\hat{b} \in B$ such that

$$\mathbf{h}^A(\hat{a}) + \mathbf{h}^B(\hat{b}) < \mathbf{h}^A(a) + \mathbf{h}^B(b) \ \text{ for all } \ (a, b) \neq (\hat{a}, \hat{b}).$$

In particular, by evaluating the inequality shown above at the point $(a, \hat{b})$ with $a \neq \hat{a}$, we have $\mathbf{h}^A(\hat{a}) < \mathbf{h}^A(a)$. If we consider the point $(\hat{a}, b)$ with $b \neq \hat{b}$ instead, we have $\mathbf{h}^B(\hat{b}) < \mathbf{h}^B(b)$. □

For $1 < p \leq \infty$ the situation is quite different.

Proposition 4.8. *Consider the metric space X_p for $1 < p \leq \infty$. Assume that B is unbounded and has a metric functional $\mathbf{h}^B = \lim_n \mathbf{h}_n^B$, where $\mathbf{h}_n^B(b) = d_B(b, b_n) - d_B(b_0, b_n)$ with $d_B(b_0, b_n) \to \infty$. Then, the mapping $(a, b) \mapsto \mathbf{h}^B(b)$ is a metric functional on X_p.*

Proof. Let us fix $a_0 \in A$ and consider internal metric functionals on X_p of the form

$$\mathbf{h}_{(a_0, b_n)}(a, b) = d_p((a, b), (a_0, b_n)) - d_p((a_0, b_0), (a_0, b_n)).$$

Then, for the case $1 < p < \infty$ we have

$$\mathbf{h}_{(a_0, b_n)}(a, b) = \mathbf{h}^B(b) + o(1) \ \text{ as } \ d_B(b_0, b_n) \to \infty,$$

and for the case $p = \infty$ we have

$$\mathbf{h}_{(a_0, b_n)}(a, b) = \max\{d_A(a, a_0) - d_B(b_0, b_n),\, \mathbf{h}_n^B(b)\}.$$

From the formulas shown above we conclude that $\mathbf{h}_{(a_0, b_n)}(a, b) \to \mathbf{h}^B(b)$ as $d_B(b_0, b_n) \to \infty$. □

Example 4.9. Consider the space $X_p = \ell_1 \oplus \ell_2$, where ℓ_1 and ℓ_2 are equipped with their usual norms. Then, X_1 has ZFP while X_p with $1 < p \leq \infty$ does not.

We will hereinafter consider properties of metric functionals on subsets X of Banach spaces E. The standard notations B_E and S_E will denote respectively the closed unit ball of E and the unit sphere of E.

Proposition 4.10. *If a Banach space E has ZFP then $X = \delta B_E$ has ZFP for all positive real numbers δ.*

Proof. This follows from the compactness of $E^{\diamondsuit}$ and the scaling property of the linear space E. Concretely, let us assume that $X = \delta B_E$ does not have ZFP for some $\delta > 0$. Then there exists a metric functional $\mathbf{h} \in X^{\diamondsuit}$ that vanishes identically in X. We know that there is a net (a_α) in X such that $\mathbf{h}(v) = \lim_\alpha \mathbf{h}_\alpha(v)$, where $\mathbf{h}_\alpha(v) = \|v - a_\alpha\| - \|a_\alpha\|$ for all $v \in E$. Now, for each integer $m \geq 1$ let us consider the internal metric functionals

$$\mathbf{h}_\alpha^m(v) = \|v - ma_\alpha\| - \|ma_\alpha\| = m\,\mathbf{h}_\alpha(m^{-1}v).$$

For fixed $m \geq 1$ and $v \in E$, the limit $\lim_\alpha \mathbf{h}_\alpha^m(v)$ exists and equals $m\,\mathbf{h}(m^{-1}v)$. By compactness of $E^{\diamondsuit}$, the aforementioned limit determines a metric functional $\mathbf{h}^m \in E^{\diamondsuit}$ that vanishes identically in the ball mX. Thus, for all $v \in E$ we have $\mathbf{h}^m(v) \to 0$ as $m \to \infty$. This limit determines precisely the metric functional vanishing identically in the whole E, because $E^{\diamondsuit}$ is compact. $\square$

A simple modification of the previous proof gives the following.

Proposition 4.11. *Assume that X is a cone of a Banach space E, meaning $tx \in X$ for all $x \in X$ and for all $t > 0$. If there exists a metric functional $\mathbf{h} \in X^{\diamondsuit}$ vanishing identically in $X \cap \delta B_E$ for some $\delta > 0$, then X does not have ZFP.*

Before we state more properties of metric functionals, let us recall some standard notations used in Banach space theory. For a given Banach space E we will denote by E^* the set of all continuous linear functionals on E. We know that E^* becomes a Banach space itself when we equip it with the operator norm. To simplify our exposition, we will use the same notation to denote both the norm on E and the operator norm on E^*. For a nonzero vector $x \in E$ we denote by $\partial\|x\|$ the set of subdifferentials:

$$\partial\|x\| = \{f \in E^* \mid \langle x, f\rangle = \|x\|,\ \|f\| = 1\}$$

and by $j(x)$ the set of dual vectors:

$$j(x) = \{f \in E^* \mid \langle x, f\rangle = \|x\|^2,\ \|f\| = \|x\|\}.$$

We show below that every subset of a Banach space that contains a ray has a metric functional without lower bounds. In this context, a metric functional associated with a ray is sometimes called a *Busemann function.*

Proposition 4.12. *Let X be a subset of a Banach space E such that there exists a vector $u \in S_E$ with the property that $tu \in X$ for all $t \geq 0$. Then, for all $x \in E$ we have*

$$\lim_{t\to\infty}(\|x - tu\| - t) = \max\{-\langle x, f\rangle \mid f \in \partial\|u\|\}.$$

This limit determines a metric functional $\mathbf{h} \in X^\diamondsuit$ such that $\mathbf{h}(su) = -s$ for all $s \geq 0$.

Proof. This follows immediately from the definition of the set $\partial\|u\|$. In fact, if we fix $x \in E$ then for all $f \in \partial\|u\|$ and for all $t > 0$ we have

$$\|tu - x\| \geq \|tu\| + \langle -x, f\rangle. \tag{4.3}$$

The limit exists as it takes place in a two-dimensional subspace spanned by u and x. We notice also that the real-valued function $t \mapsto \|x - tu\| - \|tu\|$ is non-increasing. As the inequality (4.3) holds for all f in the norm-closed set $\partial\|u\|$, the maximum is obtained in the limit. □

Example 4.13. Let us consider $X = \{ne_n\}_{n\in\mathbb{Z}}$ as a subset of the Banach space c_0, both equipped with the sup-norm. Thus, X is unbounded but does not contain a ray. We notice that each metric functional $\mathbf{h} \in X^\diamondsuit$ is either internal or the zero functional.

Remark 4.14. If the unit sphere S_E is smooth, the set $\partial\|u\|$ contains only one element, say u^*. The corresponding metric functional in Proposition 4.12 becomes

$$\mathbf{h}(x) = -\langle x, u^*\rangle.$$

Example 4.15. Let us equip $\mathbb{R}^2$ with the norm $\|x\| = |x_1| + |x_2|$ and fix the point $u = (1, 0)$ to determine the ray so that $\partial\|u\| = \{(1, \eta) \mid |\eta| \leq 1\}$ and $\max_{|\eta|\leq 1}(-x_1 - \eta x_2) = -x_1 + |x_2|$. If we perturb the unit ball a little so that it becomes uniformly convex but still has corners with a little bit larger angle, then with some $0 < \delta < 1$, we obtain likewise $\mathbf{h}(x) = -x_1 + \delta|x_2|$.

The following result was shown in [13, Section 5]. We present here a different proof.

Theorem 4.16. *Let E be a uniformly smooth Banach space. If $\mathbf{h} \in E^\diamondsuit$ is a metric functional that arises from a net (a_α) in E when $\|a_\alpha\| \to \infty$, then there exists $u^* \in B_{E^*}$ such that $\mathbf{h}(x) = \langle x, u^* \rangle$ for all $x \in E$. Conversely, if $u^* \in B_{E^*}$ is given then there exists a sequence (a_n) in E such that for all $x \in E$ we have $\mathbf{h}_{a_n}(x) \to \langle x, u^* \rangle$.*

Proof. Since E is uniformly smooth, the norm of E has the property:

$$\lim_{t \to 0} \frac{\|u + tv\| - \|u\| - \langle tv, j(u) \rangle}{\|tv\|} = 0$$

uniformly for all $u, v \in S_E$, where $j(u)$ is the unique dual vector of u (see [21, p. 36]). Let us assume that $a_\alpha \neq 0$ for all α and let $u_\alpha = a_\alpha / \|a_\alpha\|$. Now, we fix $x \in E$ and let $tv = -x/\|a_\alpha\|$. We notice that

$$\mathbf{h}_\alpha(x) = \|x - a_\alpha\| - \|a_\alpha\| = \|a_\alpha\|(\|u_\alpha + tv\| - 1).$$

Thus, we have

$$\mathbf{h}_\alpha(x) = \|a_\alpha\|\big(\langle tv, j(u_\alpha) \rangle + \|tv\| o(1)\big) = -\langle x, j(u_\alpha) \rangle + \|x\| o(1)$$

as $\|a_\alpha\| \to \infty$. By the Banach-Alaoglu theorem, the net $(j(u_\alpha))$ has a limit point $-u^* \in B_{E^*}$, and hence $\mathbf{h}(x) = \langle x, u^* \rangle$.

Conversely, let us assume that $u^* \in B_{E^*}$ is given. We choose any sequence (f_n^*) in S_{E^*} converging weakly to 0 and scalars t_n so that $\|u^* + t_n f_n^*\| = 1$. Let $u_n \in S_E$ be such that $j(u_n) = u^* + t_n f_n^*$ and let $a_n = n u_n$. Then, $\mathbf{h}_{a_n}(x)$ converges to $\langle x, u^* \rangle$ for all $x \in E$. $\square$

As seen from Example 4.15, the nonsmooth points on the unit sphere S_E create metric functionals which are not linear. Likewise, all linear functionals of norm at most 1 need not be metric functionals. By Theorem 4.2, every metric functional on ℓ_1 is linear if and only if it has the form $\mathbf{h}(x) = \sum_{j=1}^{\infty} \varepsilon_j x_j$ where $\varepsilon_j \in \{-1, 1\}$ for all $j \in \mathbb{N}$. Therefore, ℓ_1 does have ZFP.

Karlsson showed that ℓ_∞ has ZFP (see [7, Proposition 22]). Below we give a different proof.

Proposition 4.17. *Let S be a nonempty set. Assume that $\mathcal{B}(S)$, the vector space of real-valued bounded functions on S, is equipped with the sup-norm. Then, $\mathcal{B}(S)$ has ZFP.*

Proof. Let us consider a metric functional $\mathbf{h} \in \mathcal{B}(S)^\diamondsuit$ and assume that (a_α) is a net in $\mathcal{B}(S)$ such that $\mathbf{h} = \lim_\alpha \mathbf{h}_\alpha$, where $\mathbf{h}_\alpha(x) = \|x - a_\alpha\| - \|a_\alpha\|$ for all $x \in \mathcal{B}(S)$. Now, let (δ_α) be a net of positive reals converging to 0. Then, for each α there exists a point $s_\alpha \in S$ such that

$$\|a_\alpha\| \leq |a_\alpha(s_\alpha)| + \delta_\alpha.$$

Suppose that (a_α) has a subnet (a_β) for which $a_\beta(s_\beta) > 0$. Let $u \in \mathcal{B}(S)$ be the constant function $u(s) = -1$ for all $s \in S$. We notice that

$$\|u - a_\beta\| \geq 1 + a_\beta(s_\beta) \geq 1 + \|a_\beta\| - \delta_\beta.$$

Thus, we have $1 - \delta_\beta \leq \mathbf{h}_\beta(u) \leq 1$, and hence $\mathbf{h}(u) = 1$. If such subnet (a_β) does not exist, consider $u = 1$ instead. $\square$

By an argument quite similar to the one used in the previous proof, we can show that the Banach space $\mathcal{C}(K)$ of continuous functions on a compact Hausdorff space K has ZFP.

Proposition 4.18. *Let K be a compact Hausdorff space and let $u(s) = 1$ for all $s \in K$. Then, for all $\mathbf{h} \in \mathcal{C}(K)^\diamondsuit$ and for all $t \geq 0$ we have*

$$\mathbf{h}(tu) = t \quad \text{or} \quad \mathbf{h}(-tu) = t.$$

In particular, $\mathcal{C}(K)$ has ZFP.

As any normed space E has metric functionals which are not bounded below, the question of $X \subset E$ having UMP is interesting essentially only when X is bounded. We shall formulate the observations for $X = B_E$, the closed unit ball of E.

The first observation is that B_{c_0} does not have ZFP. Hence, additional assumptions are needed on E such that B_E would have ZFP or UMP. We consider three different properties of E which guarantee that B_E has UMP or at least ZFP.

Definition 4.19. *A Banach space E has the Radon-Riesz property if weak convergence and convergence of the norms imply strong convergence: $a_n \to a$ weakly and $\|a_n\| \to \|a\|$ imply $\|a_n - a\| \to 0$.*

For example, every uniformly convex Banach space has the Radon-Riesz property.

Proposition 4.20. *Let E be a Banach space with the Radon-Riesz property. If the closed unit ball B_E is weakly compact, then B_E has ZFP.*

Proof. Let $\mathbf{h}$ be an element of $(B_E)^\diamondsuit$. Let us assume that (a_α) is a net of vectors in B_E such that for all $x \in E$ we have $\mathbf{h}_\alpha(x) \to \mathbf{h}(x)$, where $\mathbf{h}_\alpha(x) = \|x - a_\alpha\| - \|a_\alpha\|$. Since $(\|a_\alpha\|)$ is a net of real numbers in the closed interval $[0,1]$, there is a subsequence (a_{α_n}) such that $\|a_{\alpha_n}\| \to r$ where $\|a\| \le r \le 1$. As B_E is weakly compact, the Eberlein-Šmulian theorem [22] implies the existence of a subsequence, denoted again by (a_{α_n}), that converges weakly to some $a \in B_E$.

Let us assume that $\|a\|$ is positive and consider the vector $y = -\frac{1}{\|a\|}a$ in B_E. Then, we have $\|y-a\| = 1+\|a\|$. If we denote by f the linear functional $\frac{1}{1+\|a\|}j(y-a)$ in B_{E^*}, we have

$$\|y - a_{\alpha_n}\| - \|a_{\alpha_n}\| \ge \langle y - a_{\alpha_n}, f\rangle - \|a_{\alpha_n}\|.$$

From the inequality shown above we get $\mathbf{h}(y) \ge \|y-a\|-r = 1+\|a\|-r > 0$.

Now, let us assume that $\|a\|$ equals 0. We may assume that r is positive, otherwise we have $\mathbf{h}(x) = \|x\|$ for all x. Let us choose a vector z such that $\|z\| = r$ and assume that $\|z - a_{\alpha_n}\| \to r$. The Radon-Riesz property implies that $(z - a_{\alpha_n})$ converges strongly to z, that is, (a_{α_n}) converges strongly to 0, which is a contradiction. Therefore, we have $\mathbf{h}(z) \ne 0$. $\square$

Definition 4.21. *A Banach space E has the Opial property, if whenever a sequence (a_n) in E converges weakly to a, the following holds:*

$$\liminf\|a_n - a\| < \liminf\|a_n - x\| \quad \textit{for all} \quad x \ne a.$$

All the ℓ_p spaces with $1 \le p < \infty$ have the Opial property. A Banach space has the Opial property if and only if whenever (a_n) converges weakly to a nonzero limit a, the number $\liminf_n\langle a, a_n^*\rangle$ is positive, where $a_n^* \in j(a_n)$ (see [23, Theorem 1]).

Proposition 4.22. *Let E be a Banach space with the Opial property. If X is a nonempty weakly compact subset of E, then X has UMP.*

Proof. Let $\mathbf{h}$ be an element of $X^\diamondsuit$. Let us assume that (a_α) is a net in X such that for all $x \in X$ we have $\mathbf{h}(x) = \lim_\alpha \mathbf{h}_\alpha(x)$, where $\mathbf{h}_\alpha(x) = \|x-a_\alpha\|-\|a_\alpha\|$. As X is weakly compact, there is a subsequence (a_{α_n}), a vector $a \in X$, and a nonnegative real number r such that $a_{\alpha_n} \to a$ weakly and $\|a_{\alpha_n}\| \to r$. Due to the Opial property, for all $x \in X \setminus \{a\}$ we have

$$\mathbf{h}(x) = \lim_n \|x - a_{\alpha_n}\| - r \; > \; \lim_n \|a - a_{\alpha_n}\| - r \; = \; \mathbf{h}(a).$$

$\square$

Another important concept is the following.

Definition 4.23. *([24], [25, Definition 3.1]) Let (a_n) be a sequence in a Banach space E. If there exists a vector $a \in E$ such that for all $x \in E$ we have*

$$\|a_n - a\| \leq \|a_n - x\| + o(1)$$

when $n \to \infty$, then one says that (a_n) is $\Delta-$convergent to a.

Weak convergence and Δ-convergence coincide on Banach spaces that have the Opial property and are both uniformly smooth and uniformly convex (see [25, Theorem 3.19]). This is true in particular on all Hilbert spaces and all the ℓ_p spaces with $1 < p < \infty$. A key result for Δ-convergence is the following theorem, which has a resemblance to the Banach-Alaoglu theorem.

Theorem 4.24. ([24, Theorem 4]) *Let E be a uniformly convex Banach space. Then, every bounded sequence (a_n) in E has a Δ-convergent subsequence.*

We have immediately the following.

Proposition 4.25. *Let X be a bounded subset of a uniformly convex Banach space E. If (a_n) is a sequence in X such that $(\mathbf{h}_{a_n})$ converges to a metric functional $\mathbf{h} \in X^\diamondsuit$, then there exists a unique vector $a \in E$ such that for all $x \in E$ we have $\mathbf{h}(a) \leq \mathbf{h}(x)$.*

Proof. By Theorem 4.24, the sequence (a_n) has a subsequence (a_{n_k}) that is Δ-convergent to some vector $a \in E$. By taking a suitable subsequence, we may assume that $\|a_{n_k}\| \to r$. Then, for all $x \in E$ the limit $\lim_k \|x - a_{n_k}\|$ exists and equals $\mathbf{h}(x) + r$. By Δ-convergence, we have $\mathbf{h}(a) \leq \mathbf{h}(x)$ for all $x \in E$. Now, let us assume that there is a vector $b \in E$ such that $b \neq a$ and $\mathbf{h}(a) = \mathbf{h}(b)$. Since $\mathbf{h}$ is a convex functional (see [4, Proposition 10]), we have $\mathbf{h}(\frac{1}{2}(a + b)) = \mathbf{h}(a)$. Thus, we have

$$\lim_k \|a - a_{n_k}\| = \lim_k \|b - a_{n_k}\| = \lim_k \|(1/2)(a + b) - a_{n_k}\| = \mathbf{h}(a) + r.$$

Since we assumed that $b \neq a$, the number $M = \mathbf{h}(a) + r$ must be positive. Since E is uniformly convex, we must have

$$\lim_k \|M^{-1}(a - a_{n_k}) - M^{-1}(b - a_{n_k})\| = 0.$$

Thus, we have $M^{-1}\|a - b\| = 0$, which is a contradiction. □

5. Proofs of the main results

Proof of Theorem 3.1. Let T and U be two elements of $\mathcal{F}$. That is, both T and U are affine nonexpansive mappings from E to itself and for all $x \in E$ we have $TUx = UTx$. Let us choose an arbitrary vector $w \in X$. For each positive integer n let us consider the vectors $a_n = n^{-1}(w + Tw + \cdots + T^{n-1}w)$ and $b_n = n^{-1}(a_n + Ua_n + \cdots + U^{n-1}a_n)$. Since both T and U map the convex set X into itself, both (a_n) and (b_n) are sequences in X such that

$$\|b_n - Tb_n\| \leq \|a_n - Ta_n\| \leq n^{-1}\|w - T^n w\|$$

and

$$\|b_n - Ub_n\| \leq n^{-1}\|a_n - U^n a_n\| \leq n^{-1}\|w - U^n w\|.$$

Due to Remark 3.2, our assumption $m(T, X) = m(U, X) = 0$ implies that $\|b_n - Tb_n\| \to 0$ and $\|b_n - Ub_n\| \to 0$ when $n \to \infty$. Now, for each $n \geq 1$ let $\mathbf{h}_n \in E^{\diamondsuit}$ denote the metric functional defined for all $x \in E$ by the formula $\mathbf{h}_n(x) = \|x - b_n\| - \|b_n\|$. Next, we notice that for all $x \in E$ we have

$$\mathbf{h}_n(Tx) - \mathbf{h}_n(x) \leq \|b_n - Tb_n\|$$

and

$$\mathbf{h}_n(Ux) - \mathbf{h}_n(x) \leq \|b_n - Ub_n\|.$$

The compactness of $E^{\diamondsuit}$ implies that $(\mathbf{h}_n)$ has a limit point $\mathbf{h} \in E^{\diamondsuit}$ such that $\mathbf{h}(Tx) \leq \mathbf{h}(x)$ and $\mathbf{h}(Ux) \leq \mathbf{h}(x)$.

We have so far proved our theorem for two elements of the family $\mathcal{F}$. Our claim is in fact true for all finite subsets of the family $\mathcal{F}$, as a quick inspection of the previous procedure reveals. With that fact in mind, we now proceed to prove the general case. For each $T \in \mathcal{F}$ let M_T denote the set of all metric functionals $\mathbf{h} \in E^{\diamondsuit}$ such that $\mathbf{h}(Tx) \leq \mathbf{h}(x)$ for all $x \in E$. We notice that each M_T is a nonempty closed subset of $E^{\diamondsuit}$ and the family $\{M_T \mid T \in \mathcal{F}\}$ has the finite intersection property. Since $E^{\diamondsuit}$ is compact, the set $\bigcap_{T \in \mathcal{F}} M_T$ is nonempty. This set contains a metric functional $\mathbf{h}$ that is subinvariant for all $T \in \mathcal{F}$.

Proof of Theorem 3.3. Let us assume that T and U are two elements of the family $\mathcal{F}$. For each positive integer n let x_n denote the vector $T^n U^n x_0$. As both T and U map X into X and $x_0 \in X$, (x_n) is a sequence in X. Now, let $(\mathbf{h}_n)$ be the sequence in $E^\diamondsuit$ defined by the formula $\mathbf{h}_n(x) = \|x - x_n\| - \|x_n\|$ for all $x \in E$. Since T and U are defined on the whole space E, for all $x \in E$ and for all $n \geq 1$ we have

$$\mathbf{h}_n(Tx) - \mathbf{h}_n(x) \leq \|x_n - Tx_n\| \leq \|T^n x_0 - T^{n+1} x_0\|$$

and

$$\mathbf{h}_n(Ux) - \mathbf{h}_n(x) \leq \|x_n - Ux_n\| \leq \|U^n x_0 - U^{n+1} x_0\|.$$

In the two previous inequalities each term situated farthest to the right is assumed to converge to 0 when $n \to \infty$. Therefore, the compactness of $E^\diamondsuit$ implies that $(\mathbf{h}_n)$ has a limit point $\mathbf{h} \in E^\diamondsuit$ such that $\mathbf{h}(Tx) \leq \mathbf{h}(x)$ and $\mathbf{h}(Ux) \leq \mathbf{h}(x)$. The rest of the proof follows from a compactness argument similar to the one used in the proof of Theorem 3.1.

Proof of Theorem 3.5. Let us assume that $\mathbf{h} \in X^\diamondsuit$ is subinvariant for the mapping $T : X \to X$. That is, for all $x \in X$ we have $\mathbf{h}(Tx) \leq \mathbf{h}(x)$. We know that the space X has UMP due to Proposition 4.22. Thus, there is a vector $a \in X$ such that for all $x \in X \setminus \{a\}$ we have $\mathbf{h}(a) < \mathbf{h}(x)$. The preceding two inequalities imply that $Ta = a$.

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