Metric functionals and weak convergence

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Abstract

We introduce a notion of weak convergence in arbitrary metric spaces. Metric functionals are key in our analysis: weak convergence of sequences in a given metric space is tested against all the metric functionals defined on said space. When restricted to bounded sequences in normed linear spaces, we prove that our notion of weak convergence agrees with the standard one.

1 Introduction

A natural question to ask is whether there exists a notion of weak convergence which may be valid in *all* metric spaces. If such a concept exists, one may naturally expect that it agrees with the standard weak convergence in all normed linear spaces.

While investigating certain fixed point problems, T.-C. Lim [Lim77] introduced an interesting notion of convergence in metric spaces, the so-called $\Delta-$convergence. This concept, however, does not agree with the standard weak convergence in *some* normed linear spaces. This is discussed in the end of Section 3.

Lim's work has nevertheless motivated some investigations on weak topologies in CAT(0) spaces. In these spaces, one can define metric projections on compact geodesics. This has led researchers to consider a notion of weak

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convergence that agrees with the standard one in Hilbert spaces. See the works [KP08], [EFL09], [Bač23], and [LP23].

In this note, we present a novel notion of weak convergence in metric spaces. Our approach uses metric functionals.

Definition 1.1. Let (X, d) be a metric space. We say that a sequence of points $(x_n)_{n\geq 1}$ in X converges $d-$*weakly* to a point z in X if we have

$$\liminf_{n\to\infty} \mathbf{h}(x_n) \geq \mathbf{h}(z), \tag{1.1}$$

for all metric functionals $\mathbf{h}$ on X. We use the notation $x_n \overset{d}{\rightharpoonup} z$.

Our concept of $d-$weak convergence can be applied to nets. We discuss here $d-$weak convergence of sequences only to simplify the exposition of our ideas.

We also want to emphasize that $d-$weak convergence fully depends on the given metric d. The metric d determines all the corresponding metric functionals $\mathbf{h}$ that are essential to test the inequality (1.1).

The concept of a metric functional is fairly known in some mathematical disciplines. Nonetheless, we recall such a concept and provide new properties in Section 2.

We proceed now to provide evidence that supports our choice of Definition 1.1 as a suitable candidate for weak convergence in metric spaces. The following nontrivial statements concern $d-$weak convergence of sequences in normed linear spaces.

Theorem 1.2. *Let $(X, \|\cdot\|)$ be a normed linear space. Let d be the metric on X induced by the norm $\|\cdot\|$. If a sequence of points $(x_n)_{n\geq 1}$ in X converges $d-$weakly, then its limit is unique.*

The following result shows the connection of $d-$weak convergence with the standard weak convergence in normed linear spaces. Namely, they agree for bounded sequences in *all* normed linear spaces.

Theorem 1.3. *Let $(X, \|\cdot\|)$ be a normed linear space. Let d be the metric on X induced by the norm $\|\cdot\|$. Suppose that $(x_n)_{n\geq 1}$ is a bounded sequence in X and z is a point in X. Then, $(x_n)_{n\geq 1}$ converges $d-$weakly to z if and only if $(x_n)_{n\geq 1}$ converges weakly to z.*

One may wonder whether unbounded sequences in normed linear spaces converge $d-$weakly. We can prove that this does not happen in some normed linear spaces.

Theorem 1.4. *Suppose that X is ℓ_1, or $C[0,1]$, or a normed linear space whose dual is strictly convex. If d is the corresponding induced metric, then unbounded sequences in X do not converge $d-$weakly.*

Conjecture 1.5. *There is no normed linear space where unbounded sequences converge $d-$weakly.*

Without a proof of the above conjecture, it is unclear how $d-$weak convergence behaves on linear combinations of two $d-$weakly convergent sequences. Nevertheless, we can offer the following.

Theorem 1.6. *Let $(X, \|\cdot\|)$ be a normed linear space. Let d be the metric on X induced by the norm $\|\cdot\|$. Suppose that $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ are two sequences in X. If there are two vectors u and v in X such that $x_n \xrightarrow{d} u$ and $d(y_n, v) \to 0$, then we have*

$$sx_n + ty_n \xrightarrow{d} su + tv,$$

for all real numbers s and t.

All the results presented above examine $d-$weak convergence of sequences in the whole normed linear space. Proper subsets become legit metric spaces when we equip them with the subspace topology. With this in mind, we can produce simple examples of metric spaces where unbounded sequences converge $d-$weakly, or where $d-$weakly convergent sequences have more than one limit. An example where both situations happen is the following. Equip the set $X = \{0, e_1, 2e_2, 3e_3, \dots\}$ with the metric d that is induced by the norm of ℓ_1. We can verify that the unbounded sequence $(ne_n)_{n\geq 1}$ in X converges $d-$weakly to every point in X. More examples are presented in Section 3.

The following result states that the set of all $d-$weak limits of a sequence in a $W-$convex metric space is also $W-$convex. The concept of $W-$convexity was introduced by W. Takahashi in [Tak70]. We recall this concept in Definition 2.7. Briefly we mention that every metric space that admits a conical geodesic bicombing is $W-$convex.

Theorem 1.7. *Let (X, d) be a $W-$convex metric space. Then, the set of all $d-$weak limits of a sequence in X is $W-$convex and closed.*

2 Metric functionals

2.1 Where do they appear?

Metric functionals have become valuable tools in investigations of iterative processes in metric structures.

Fixed point problems in metric spaces are studied by means of metric functionals. Briefly, instead of looking directly for a fixed point in the space, one rather seeks a metric functional that remains (sub)invariant by the iterations of the mapping or family of mappings involved in the problem. These ideas have been put into action in [GV12], [LLN18], [GK21], [Lin22], [Kar24], [GN24].

Metric functionals also have been applied to study noncommuting random products. There is always a metric functional that determines the direction along which a random product grows [GK20]. Motivated by this result, A. Avelin and A. Karlsson [AK22] have used metric functionals to study a cut-off phenomenon associated with the dynamics of deep neural networks.

Metric functionals have appeared recently in studies on Lipschitz-free spaces [Meg24] and Hermitian symmetric spaces [CCAL24].

2.2 What are they like?

Let (X, d) be a metric space. We choose a basepoint o in X and for each point w in X we consider the funtional

$$\mathbf{h}_w(x) = d(x, w) - d(o, w), \text{ for all } x \text{ in } X. \tag{2.1}$$

We equipped the product space $\mathbb{R}^X$ with the topology of pointwise convergence and denote by $X^\vee$ the subset of $\mathbb{R}^X$ containing all the functionals of the form (2.1). The closure of $X^\vee$ is denoted by $X^\diamondsuit$ and each element $\mathbf{h}$ in it is called a *metric functional*.

The next statement is obtained by applying properties of the metric d to the functionals (2.1).

Proposition 2.1 ([Gut19a, Chapter 3]). *The following properties hold:*

1. *The mapping $w \mapsto \mathbf{h}_w$ from X to $X^\vee$ is injective and continuous.*

2. *The space $X^\diamondsuit$ is compact and Hausdorff. In particular, for every metric functional $\mathbf{h} \in X^\diamondsuit$ there exists a net of points (w_α) in X such that*

$$\mathbf{h}(x) = \lim_\alpha \mathbf{h}_{w_\alpha}(x), \text{ for all } x \text{ in } X.$$

3. *Every metric functional $\mathbf{h} \in X^\diamondsuit$ is $1-$Lipschitz; that is, we have*

$$|\mathbf{h}(x) - \mathbf{h}(y)| \leq d(x, y), \text{ for all } x \text{ and } y \text{ in } X.$$

Remark 2.2. The set $X^\diamondsuit$ is sometimes called the *metric compactification* of X. The spaces X and $X^\vee$, however, need not be homeomorphic. In fact, for the unbounded sequence $(ne_n)_{n\geq 1}$ in $X = \ell_1$ we have $\mathbf{h}_{ne_n} \to \mathbf{h}_0$. Another example is the CAT(0) space given in [Kar22, Example 3.2]. If one insists on having $X^\diamondsuit$ as a standard topological compactification of X, some conditions on X must hold. Such conditions are discussed in [DGJTG24, Theorem 2.1].

The existence of the continuous injection $X \hookrightarrow X^\vee \subset X^\diamondsuit$ motivates us to call all the elements in $X^\vee$ *internals*. In this way, an analog of the Banach-Alaoglu theorem reads: *Given a net of internals* $(\mathbf{h}_{w_\alpha})$*, there exists a subnet* $(\mathbf{h}_{w_\beta})$ *and a metric functional* $\mathbf{h} \in X^\diamondsuit$ *such that* $\mathbf{h}_{w_\beta}(x) \to \mathbf{h}(x)$ *for all* x *in* X. This type of convergence may be seen as an analog of the weak-star convergence of continuous linear functionals. Notice that all metric functionals are continuous. In the linear theory, however, continuity of linear functionals must be indicated explicitly.

There exist explicit formulas for all metric functionals on infinite dimensional ℓ_p spaces with $1 \leq p < \infty$, see [Gut19b]. Explicit formulas for an important class of metric functionals on $C(K)$ are shown in [Wal18].

Remark 2.3. Limits of the type $\mathbf{h}_{w_n} \to \mathbf{h}$ behave differently in ℓ_1 and ℓ_p with $1 < p < \infty$. The functional identically zero, denoted by $\mathbf{0}$, is a metric functional on each ℓ_p with $1 < p < \infty$, but is not on ℓ_1. Define $w_n := ne_n$ for all $n \geq 1$. In ℓ_p we have $\mathbf{h}_{w_n} \to \mathbf{0}$, whereas in ℓ_1 we have $\mathbf{h}_{w_n} \to \mathbf{h}_0$.

2.3 What properties do they have?

The space $X^\diamondsuit$ is not always metrizable. It becomes metrizable when X is separable. To be precise, we have the following.

Proposition 2.4. *Let* (X, d) *be a separable metric space. Then, every sequence of points* $(x_n)_{n\geq 1}$ *in* X *has a subsequence* $(w_i)_{i\geq 1}$ *such that the sequence of internals* $(\mathbf{h}_{w_i})_{i\geq 1}$ *converges in* $X^\diamondsuit$.

Proof. We fix a point o in X and consider a countable dense subset $\{y_1, y_2, ...\}$ of X. Let $(x_n)_{n\geq 1}$ be a sequence in X. Each internal $\mathbf{h}_{x_n}$ satisfies the property $|\mathbf{h}_{x_n}(y_k)| \leq d(o, y_k)$ for all positive integers k. By using Cantor's diagonal argument, we obtain a subsequence $(w_i)_{i\geq 1}$ of $(x_n)_{n\geq 1}$ such that for every positive integer k the sequence $(\mathbf{h}_{w_i}(y_k))_{i\geq 1}$ converges to a real number, say r_k, in the closed interval $[-d(o, y_k),\, d(o, y_k)]$. Now, since $X^\diamondsuit$ is a compact Hausdorff space, the sequence of internals $(\mathbf{h}_{w_i})_{i\geq 1}$ has a limit point $\mathbf{h} \in X^\diamondsuit$ so that $\mathbf{h}(y_k) = r_k$ for all k.

Let x be a point in X and ϵ be a positive real number. Then, there is an integer $m \geq 1$ such that $d(x, y_m) < \epsilon/4$. The identity

$$\lim_{i\to\infty} \mathbf{h}_{w_i}(y_m) = \mathbf{h}(y_m)$$

implies that there is an integer $i_0 \geq 1$ such that $|\mathbf{h}_{w_i}(y_m) - \mathbf{h}(y_m)| < \epsilon/2$ for all $i \geq i_0$. Thus, for every $i \geq i_0$ we have

$$|\mathbf{h}_{w_i}(x) - \mathbf{h}(x)| \leq 2\, d(x, y_m) + |\mathbf{h}_{w_i}(y_m) - \mathbf{h}(y_m)| < \epsilon\,.$$

□

Sometimes, the space $X^\diamondsuit$ consists of only internals. For example, if X is a compact metric space, then we have $X^\diamondsuit = X^\vee$. The same phenomenon can occur for noncompact sets; for example when X is the closed unit ball of ℓ_1.

There are metric spaces X for which we have $X^\diamondsuit = X^\vee \cup \{\mathbf{0}\}$, where $\mathbf{0}$ is the functional identically zero. For example, let X be the set of real numbers and equip it with the metric $d = |\cdot - \cdot|^p$ with $0 < p < 1$.

The following property holds in all metric spaces, and is a straightforward consequence of the definition of metric functionals.

Proposition 2.5. *Let (X, d) be a metric space. Then, for every point w in X we have*

$$d(o, w) = \max_{\mathbf{h}\in X^\diamondsuit} \mathbf{h}(w).$$

Proof. The claim is trivial when w is the basepoint o because at this point every metric functional vanishes. So, let w be a point in X with $w \neq o$. Since every metric functional is $1-$Lipschitz, we have $\mathbf{h}(w) \leq d(o, w)$ for all $\mathbf{h}$ in $X^\diamondsuit$, and the equality holds for the internal $\mathbf{h}_o$. □

Remark 2.6. The Hahn-Banach theorem in normed linear spaces implies that for every vector v one has $\|v\| = \max_f |f(v)|$, where the maximum is taken over all continuous linear functionals of norm at most 1. A weaker version of this will be shown in Proposition 2.12, where the Hahn-Banach theorem is not used.

Next we discuss some properties that metric functionals possess when the given metric space has a certain convex structure.

Definition 2.7. A metric space (X, d) is said to be $W-$convex if there exists a mapping W from $X \times X \times [0, 1]$ to X such that

$$d(z, W(x, y, t)) \leq (1 - t)d(z, x) + td(z, y),$$

for all points x, y, and z in X and all t in $[0, 1]$. This concept was introduced by W. Takahashi in [Tak70] to study some fixed point theorems.

Every convex subset of a normed linear space is $W-$convex as a metric space. To see this, one considers the mapping $W(x, y, t) = (1 - t)x + ty$.

Nontrivial examples of $W-$convex metric spaces appear naturally in geometric group theory. Metric spaces that admit a *conical bicombing* are such examples. A standard reference on conical bicombings is the paper [DL15]. It seems that the authors of [DL15] were unaware of Takahashi's notion of convexity.

A real-valued functional f defined on a $W-$convex metric space (X, d) is said to be $W-$convex if for every x and y in X and t in $[0, 1]$ we have

$$f(W(x, y, t)) \leq (1 - t)f(x) + tf(y).$$

Proposition 2.8. *Let (X, d) be a $W-$convex metric space. Then, every metric functional on X is $W-$convex.*

Proof. Let $\mathbf{h}$ be a metric functional on X. Suppose that (w_α) is a net of points in X so that $\mathbf{h}_{w_\alpha} \to \mathbf{h}$. Let x and y be two points in X and t in [0,1]. Since (X, d) is $W-$convex, for all points w_α we have

$$d(W(x, y, t), w_\alpha) - d(o, w_\alpha) \leq (1 - t)\,\mathbf{h}_{w_\alpha}(x) + t\,\mathbf{h}_{w_\alpha}(y).$$

Thus, we get $\mathbf{h}(W(x, y, t)) \leq (1 - t)\,\mathbf{h}(x) + t\,\mathbf{h}(y)$. □

Now we discuss properties of an important class of metric functionals in normed linear spaces. The obvious basepoint that we choose is the origin.

Definition 2.9. Let $(X, \|\cdot\|)$ be a normed linear space. A Busemann functional associated with a unit vector u in X is denoted by $\mathbf{h}^u$ and defined as

$$\mathbf{h}^u(x) := \lim_{t \to +\infty} (\|x - tu\| - t),$$

for all x in X. This limit exists for each x in X because the function $t \mapsto \|x - tu\| - t$ defined on the interval $[0, +\infty)$ is monotone non-increasing and bounded below by $-\|x\|$.

Remark 2.10. Every Busemann functional is a metric functional.

Some of the following statements are probably well-known. We recall them here for the reader's convenience.

Proposition 2.11. *Let $(X, \|\cdot\|)$ be a normed linear space. For every v in X there are two Busemann functionals $\mathbf{h}^{u_1}$ and $\mathbf{h}^{u_2}$ such that*

$$\mathbf{h}^{u_1}(v) = \|v\| = -\,\mathbf{h}^{u_2}(v).$$

Proof. The case $v = 0$ is trivial because all metric functionals vanish at 0. We assume now that $v \neq 0$. Consider the unit vector $u_1 = (-1/\|v\|)v$. Then, we have

$$\mathbf{h}^{u_1}(v) = \lim_{t \to +\infty}(\|v - tu_1\| - t) = \lim_{t \to +\infty}(|\|v\| + t| - t) = \|v\|.$$

The unit vector $u_2 = -u_1$ gives $\mathbf{h}^{u_2}(v) = -\|v\|$. □

Proposition 2.12. *Let $(X, \|\cdot\|)$ be a normed linear space. For every vector v in X we have*

$$\|v\| = \sup\{\, \mathbf{h}(v) \mid \mathbf{h} \text{ is a Busemann functional on } X\}$$

Proof. By Proposition 2.11, for every v there is a Busemann functional $\mathbf{h}^u$ on X such that $\mathbf{h}^u(v) = \|v\|$. Thus, we have

$$\sup\{\, \mathbf{h}(v) \mid \mathbf{h} \text{ is a Busemann functional on } X\} \geq \mathbf{h}^u(v) = \|v\|.$$

The other inequality follows from the fact that every Busemann functional is a metric functional. □

Remark 2.13. The previous statement is valid in geodesic metric spaces which have the property that every geodesic segment can be extended to a ray. This was observed in [Kar21, Section 6].

Proposition 2.14. *Let $(X, \|\cdot\|)$ be a normed linear space. Every Busemann functional is subadditive and positively homogeneous. In other words, if $\mathbf{h}^u$ is a Busemann functional on X, then we have*

1. $\mathbf{h}^u(x + y) \leq \mathbf{h}^u(x) + \mathbf{h}^u(y)$, *for all x and y in X, and*
2. $\mathbf{h}^u(sx) = s\,\mathbf{h}^u(x)$, *for all x in X and for all $s \geq 0$.*

Proof. By the definition of a Busemann functional $\mathbf{h}^u$, we have

$$\mathbf{h}^u(x) = \inf_{t \geq 0}(\|x - tu\| - t),$$

for all $x \in X$. To complete the proof, it suffices to notice that if x and y are vectors in X and $s > 0$, then for every $t \geq 0$ we have

$$\mathbf{h}^u(x + y) \leq \|x + y - 2t\| - 2t \leq (\|x - tu\| - t) + (\|y - tu\| - t)$$

and

$$\|sx - tu\| - t = s(\|x - (t/s)u\| - (t/s)).$$

□

It is well-known that the Hahn-Banach theorem implies that continuous linear functionals separate points in normed linear spaces. A natural question is whether metric functionals separate points. Internals trivially do this. So, we are really looking for non-internal metric functionals that separate points.

Karlsson presents a metric Hahn-Banach statement in [Kar21, Proposition 1]. But this result does not immediately give a separation property. In normed linear spaces, one only needs the definition of a Busemann functional (Definition 2.9) to obtain the following separation statement.

Theorem 2.15. *Let $(X, \|\cdot\|)$ be a normed linear space. If x and y are two distinct vectors in X, then there exists a Busemann functional $\mathbf{h}^u$ such that $\mathbf{h}^u(x) \neq \mathbf{h}^u(y)$.*

Proof. Let u be the unit vector given by $\|x - y\|u = x - y$. We note that $\|x - y - tu\| = |\|x - y\| - t|$ for all t. Then, the Busemann functional $\mathbf{h}^u$ associated with u takes the value $-\|x - y\|$ at the vector $x - y$. We are going to show that

$$\mathbf{h}^u(x) = -\|x - y\| + \mathbf{h}^u(y).$$

Indeed, subadditivity of $\mathbf{h}^u$ implies $\mathbf{h}^u(x) \leq -\|x - y\| + \mathbf{h}^u(y)$. The other inequality always holds because $\mathbf{h}^u$ is a metric functional. □

If we use the Hahn-Banach theorem, we can obtain explicit formulas for Busemann functionals. For a given unit vector u, the set of subdifferentials

$$\partial\|u\| = \{f \in X^* \mid \langle u, f \rangle = 1, \ \|f\| = 1\}$$

is a nonempty convex and closed subset of the dual space X^*. With this information available we can show the following.

Theorem 2.16 ([GN24, Proposition 4.12])**.** *Let $(X, \|\cdot\|)$ be a normed linear space. Let u be a unit vector in X. Then, the Busemann functional $\mathbf{h}^u$ associated with u has the form*

$$\mathbf{h}^u(x) = \max_{f \in \partial\|u\|} \langle -x, f \rangle, \quad \textit{for all } x \textit{ in } X.$$

3 Details of $d-$weak convergence

To come up with our concept of weak convergence proposed in Definition 1.1, we first looked at the behavior of some sequences in metric spaces where explicit formulas for all metric functionals were available. We started on our analysis in the ℓ_p spaces with $1 \leq p < \infty$, because in these spaces explicit formulas for all metric functionals were at our disposal, see [Gut19b]. So, we thought that if we were to use metric functionals to broadly test weak convergence, it should agree with the standard weak convergence. This was indeed the case as we will show in the next section.

Let us focus on our Definition 1.1. Notice that we need all the metric functionals on (X, d) to test $d-$weak convergence. It is also important to notice that only a limit inferior and an inequality appear in our notion of $d-$weak convergence. A reason for this choice is that a whole limit of the form $\mathbf{h}(x_n) \to \mathbf{h}(z)$ with internals $\mathbf{h}$ would obviously give strong limits. This was briefly discussed in [Kar21, Section 6].

After a quick look at the construction of metric functionals given in Section 2.2, one may naturally wonder whether choosing a basepoint other than o could completely modify $d-$weak convergence. We show next that the basepoint can be chosen freely.

Proposition 3.1. *Let (X, d) be a metric space. The $d-$weak convergence of sequences in X does not depend on the choice of the basepoint upon which all the metric functionals on X are built.*

Proof. We assume that the point o in X is the original basepoint as done in Section 2.2. Choose now another basepoint b in X. Let $\boldsymbol{\eta}$ be a metric functional on X that is built on the basepoint b. Then, there exists a metric functional $\mathbf{h}$ on X that is built on the basepoint o and such that

$$\mathbf{h}(x) = \boldsymbol{\eta}(x) + \mathbf{h}(b),$$

for all x in X. This follows straightforwardly from the identity

$$d(\cdot, w) - d(o, w) = d(\cdot, w) - d(b, w) + [d(b, w) - d(o, w)].$$

Now, if a sequence $(x_n)_{n\geq 1}$ in X converges $d-$weakly to a point z in X, then we have

$$\liminf_{n\to\infty} \boldsymbol{\eta}(x_n) = \liminf_{n\to\infty} \mathbf{h}(x_n) - \mathbf{h}(b) \geq \mathbf{h}(z) - \mathbf{h}(b) = \boldsymbol{\eta}(z).$$

□

Remark 3.2. Suppose that we have $x_n \xrightarrow{d} z$. If one chooses $b := z$ as the new basepoint, then $d-$weak convergence of the same sequence $(x_n)_{n\geq 1}$ reads

$$\liminf_{n\to\infty} \boldsymbol{\eta}(x_n) \geq 0,$$

for all metric functionals $\boldsymbol{\eta}$ on X, which are now built on the basepoint z.

The result that we present next follows readily from the definition of $d-$weak convergence.

Proposition 3.3. *Let (X, d) be a metric space. If a sequence $(x_n)_{n\geq 1}$ converges $d-$weakly to a point z in X, then we have*

$$d(z, w) \leq \liminf_{n\to\infty} d(x_n, w), \tag{3.1}$$

for all points w in X.

Proof. Test $x_n \xrightarrow{d} z$ against all the internal metric functionals $\mathbf{h}_w$. □

Remark 3.4. In normed linear spaces, the inequality (3.1) also holds for sequences that converge weakly (in the standard weak topology). This is shown, however, by using the Hahn-Banach theorem.

Proposition 3.5. *Let p and q be two distinct points in a metric space (X, d). Consider the sequence of points $(x_n)_{n\geq 1}$ in X with $x_{2n} = p$ and $x_{2n-1} = q$ for all $n \geq 1$. Then, the sequence $(x_n)_{n\geq 1}$ does not converge $d-$weakly.*

Proof. Suppose that the sequence $(x_n)_{n\geq 1}$ converges $d-$weakly to a point z in X. We are going to show a contradiction with the help of both internals $\mathbf{h}_p$ and $\mathbf{h}_q$. Indeed, since we have $\mathbf{h}_p(x) > \mathbf{h}_p(p)$ for all $x \neq p$, our assumption would imply $z = p$. Then, we would have $\mathbf{h}_q(x_{2n-1}) < \mathbf{h}_q(z)$ for all integers $n \geq 1$. Thus, the sequence $(x_n)_{n\geq 1}$ cannot converge to z. □

Proposition 3.6. *Let X be a set that contains infinitely many points and is equipped with the discrete metric d. Let $(x_n)_{n\geq 1}$ be a sequence in X. Then, we have the following possibilities:*

1. *The sequence $(x_n)_{n\geq 1}$ is eventually constant. In other words, the sequence converges strongly.*
2. *If there are two subsequences $(x_{m_i})_{i\geq 1}$ and $(x_{n_i})_{i\geq 1}$ where $x_{m_i} = p$ and $x_{n_i} = q$ with $p \neq q$, then $(x_n)_{n\geq 1}$ does not converge $d-$weakly.*
3. *If there exists only one point z such that $x_n = z$ happens infinitely often, then $(x_n)_{n\geq 1}$ converges $d-$weakly to z but it need not converge strongly.*
4. *If there exists no z for which $x_n = z$ infinitely many times, then $(x_n)_{n\geq 1}$ converges $d-$weakly to all points in X.*

Proof. We only need to prove the last two possibilities. This is done after noting that every metric functional on this space (X, d) is either internal or the constant functional identically zero. □

Proposition 3.7. *Let (X, d) be a metric space. Then, every closed ball*

$$B(q, r) = \{x \in X \mid d(x, q) \leq r\}$$

is $d-$weakly sequentially closed.

Proof. Suppose that $(x_n)_{n\geq 1}$ is a sequence in $B(q, r)$ and z is a point in X such that $x_n \xrightarrow{d} z$. Here $d-$weak convergence must be tested against all the metric functionals on the whole space X. We are going to show that z is in $B(q, r)$. Indeed, if the point z were not in $B(q, r)$, we would have $d(z, q) > r$. Thus, testing $d-$weak convergence against the internal $\mathbf{h}_q$ would imply the contradiction

$$r - d(o, q) \geq \liminf_{n\to\infty} \mathbf{h}_q(x_n) \geq \mathbf{h}_q(z) > r - d(o, q).$$

□

Remark 3.8. Let (X, d) be a metric space and A be a subset of X. If we define the set $hull(A)$ as the intersection of all closed balls containing A, then $hull(A)$ is $d-$weakly sequentially closed.

Example 3.9. Let X be a nonempty set equipped with the discrete metric. For a nonempty subset A of X we have

$$hull(A) = \begin{cases} A & |A| = 1 \\ X & |A| > 1. \end{cases}$$

Here $|A|$ means the number of elements of A.

Let us now discuss relations between $d-$weak convergence and strong convergence.

Proposition 3.10. *Let (X, d) be a metric space. If a sequence $(x_n)_{n\geq 1}$ in X converges strongly, then it converges $d-$weakly.*

Proof. Suppose that the sequence $(x_n)_{n\geq 1}$ converges strongly to a point z in X. Let $\mathbf{h}$ be a metric functional on X. Since all metric functionals are $1-$Lipschitz, we have $\mathbf{h}(x_n) \geq -d(x_n, z) + \mathbf{h}(z)$ for all $n \geq 1$. From this we get

$$\liminf_{n\to\infty} \mathbf{h}(x_n) \geq \mathbf{h}(z).$$

□

When does $d-$weak convergence imply strong convergence?

In general, a sequence may have more than one $d-$weak limit. The following result states that there are special cases where we have a unique $d-$weak limit which is also a strong one.

Proposition 3.11. *Suppose that (X, d) is such that all its closed balls are compact. If a bounded sequence in X converges $d-$weakly, then it does so strongly.*

Proof. Let $(x_n)_{n\geq 1}$ be a bounded sequence in X such that $x_n \xrightarrow{d} z$. Suppose that there exists $\epsilon > 0$ such that $d(x_n, z) > \epsilon$ for infinitely many n. Since all closed balls are compact, there is a subsequence $(x_{n_i})_{i\geq 1}$ and a point p in X such that $d(x_{n_i}, p) \to 0$. By testing $d-$weak convergence on the internal $\mathbf{h}_p$, we get

$$0 \geq \liminf_{n\to\infty} d(x_n, p) \geq d(z, p).$$

Thus, we have $z = p$, which contradicts the existence of such ϵ. □

A quick inspection of the explicit formulas for all metric functionals on ℓ_1 shown in [Gut19b, Theorem 3.6] reveals the following.

Proposition 3.12. *If a sequence in ℓ_1 converges $d-$weakly, then it converges strongly.*

Proof. From the formulas shown in [Gut19b, Theorem 3.6], we notice that some of the metric functionals on ℓ_1 are linear: those are of the form

$$\mathbf{h}(x) = \sum_{k\in I} \varepsilon(k)x(k), \tag{3.2}$$

where I is a nonempty subset of $\mathbb{N}$ and $\varepsilon(k) \in \{-1, 1\}$ for all k in I. Now, let $(y_n)_{n\geq 1}$ be a sequence in ℓ_1 that converges $d-$weakly to a point y in ℓ_1. Thus, for every $k \in \mathbb{N}$ we have

$$\lim_{n\to\infty} y_n(k) = y(k).$$

Define $x_n := y_n - y$ and assume that there exists a positive real number ϵ such that $\limsup_{n\to\infty}\|x_n\| > \epsilon$. What we do next is to apply the so-called gliding hump technique. We follow the exact details that appear in the book of B. Beauzamy [Bea82, pp. 117-118]. The gliding hump technique gives a sequence of positive integers $(n_p)_{p\geq 1}$ and a sequence $(c(m))_{m\geq 1}$ in $\{-1, 0, 1\}$ such that

$$\Big|\sum_{m\geq 1} c(m)x_{n_p}(m)\Big| \geq \epsilon/4, \quad \text{if} \quad p \geq 3. \tag{3.3}$$

On the other hand, if we define $I := \{k \in \mathbb{N} \mid c(k) \neq 0\}$ and $\varepsilon(k) := c(k)$ for all k in I, we obtain a metric functional $\mathbf{h}$ of the form (3.2). Clearly, $-\mathbf{h}$ is also a metric functional of the form (3.2). By testing $d-$weak convergence of $(y_n)_{n\geq 1}$ against these two metric functionals, we get

$$\lim_{n\to\infty} \sum_{m\in I} \varepsilon(m) y_{n_p}(m) = \sum_{m\in I} \varepsilon(m) y(m).$$

This contradicts the inequality (3.3). □

In uniformly convex normed linear spaces the following statement holds.

Proposition 3.13. *Let $(E, \|\cdot\|)$ be a uniformly convex normed linear space. Assume that $X = B_E$, the closed unit ball of E. If a sequence $(x_n)_{n\geq 1}$ in X is such that $\|x_n\| \to \|\hat{x}\|$ and that for all w in X we have*

$$\liminf_{n\to\infty} \mathbf{h}_w(x_n) \geq \mathbf{h}_w(\hat{x}),$$

then $\|x_n - \hat{x}\| \to 0$.

Proof. We may assume that $\hat{x} \neq 0$ as otherwise the claim is trivial. First, suppose that we have $\|x_n\| = \|\hat{x}\|$ for all $n \geq 1$, but there is a positive real number ϵ such that

$$\limsup_{n\to\infty} \|x_n - \hat{x}\| > \epsilon \|\hat{x}\|.$$

By uniform convexity, there would exist a positive real number δ such that

$$\|(x_n + \hat{x})/2\| \leq (1-\delta)\|\hat{x}\|.$$

Testing $d-$weak convergence of the sequence $(x_n)_{n\geq 1}$ with the internal $\mathbf{h}_{-\hat{x}}$ gives

$$\liminf_{n\to\infty} \|(x_n + \hat{x})/2\| \geq \|\hat{x}\|,$$

which shows that such ϵ cannot exist.

Now, if we only have $\|x_n\| \to \|\hat{x}\|$, we can move to $y_n := \frac{\|\hat{x}\|}{\|x_n\|} x_n$ so that $\|y_n - x_n\| \to 0$. By testing $d-$weak convergence of the sequence $(x_n)_{n\geq 1}$ with the internal $\mathbf{h}_{-\hat{x}}$, we get

$$\liminf_{n\to\infty} \|y_n + \hat{x}\| \geq \liminf_{n\to\infty} \Big| \|x_n + \hat{x}\| - \|x_n - y_n\| \Big| \geq 2\|\hat{x}\|.$$

Then, the limit $\|y_n - \hat{x}\| \to 0$ holds. Hence, we have $\|x_n - \hat{x}\| \to 0$. □

Remark 3.14. In general, there are more elements in $(B_E)^\diamondsuit$ than just the internals. In the previous proposition, testing $d-$weak convergence only against internals was enough. Also, the same proof holds in the whole space E.

Next, we want to discuss some behavior of $d-$weak convergence in closed balls of ℓ_2 and ℓ_1. For this purpose, we define the following.

Definition 3.15. Let (X, d) be a metric space. For a given sequence $(x_n)_{n\geq 1}$ in X we define the set

$$\Lambda_d(x_n) := \{z \in X \mid \; x_n \xrightarrow{d} z\}.$$

Example 3.16. Let X be the closed unit ball of ℓ_2. Following the construction of metric functionals on Hilbert spaces [Gut19b], we note that all metric functionals on X are of the form

$$\mathbf{h}(x) = (\|x\|^2 - 2\langle x, z\rangle + c^2)^{1/2} - c,$$

where $\|z\| \leq c \leq 1$. For all such metric functionals $\mathbf{h}$ we have

$$\sqrt{5}/2 - 1 \leq \lim_{n\to\infty} \mathbf{h}(e_n/2) \leq 1/2.$$

Thus, we have $\Lambda_d(2^{-1}e_n) = (\sqrt{5}/2 - 1)X$. In general, if we consider the sequence $(x_n)_{n\geq 1}$ in X with $x_n = \theta e_n$ and $0 < \theta \leq 1$, then we observe that a point u is in $\Lambda_d(\theta e_n)$ if and only if $\|u\|^2 + 2\|u\| \leq \theta^2$. Clearly, we cannot have $\|x_n\| \to \|u\|$.

Example 3.17. Suppose that the set $X = \{0, e_1, e_2, \cdots\}$ is equipped with the metric d induced by the norm of ℓ_1. All the metric functionals on X are internal: $\mathbf{h}_0(x)$ and $\mathbf{h}_{e_j}$ for all $j \geq 1$. Thus, we have $\Lambda_d(e_n) = X$.

Example 3.18. Let X be the closed unit ball of ℓ_1. Then, all the metric functionals on X are internal [Gut19b], and hence we have $\Lambda_d(2^{-1}e_n) = 2^{-1}X$. To see this it suffices to note that for all w in X we have

$$\mathbf{h}_w(2^{-1}e_n) = \|2^{-1}e_n - w\|_1 - \|w\|_1 \to 1/2.$$

Before we close this section we want to point out the difference between our notion of $d-$weak convergence in metric spaces and the one proposed by T. C. Lim [Lim77]; the so called $\Delta-$convergence.

A sequence $(x_n)_{n\geq 1}$ in a metric space (X, d) is said to $\Delta-$converge to a point z in X if for every y in X one has $d(z, x_n) \leq d(y, x_n) + o(1)$ as $n \to \infty$. It is known that when $X = \ell_2$ the standard weak convergence agrees with $\Delta-$convergence for bounded sequences. However, in $X = \ell_1$ these concepts do not agree. To see this, consider the sequence $(x_n)_{n\geq 1}$ in ℓ_1 where $x_n = e_n$ for all $n \geq 1$. While the sequence does not converge $d-$weakly, it $\Delta-$converges to 0. Thus, the concept of $\Delta-$convergence fails to agree with weak convergence of bounded sequences in some Banach spaces.

4 Proofs of the main theorems

Proof of Theorem 1.2. Suppose that there are two distinct vectors u and v in X for which we have

$$x_n \xrightarrow{d} u \quad \text{and} \quad x_n \xrightarrow{d} v. \tag{4.1}$$

Let B_{X^*} denote the closed unit ball of the dual space X^*. Let $\mathcal{E}(B_{X^*})$ denote the set of all extreme points of B_{X^*}. The inclusion $\mathcal{E}(B_{X^*}) \subset X^{\diamondsuit}$ holds as a result of the statement in [Wal18, Corollary 3.5]. Thus, testing $d-$weak convergence in (4.1) against every element g in $\mathcal{E}(B_{X^*})$ gives

$$\lim_{n\to\infty} \langle x_n, g\rangle = \langle u, g\rangle = \langle v, g\rangle.$$

Clearly, we have $\langle u, f\rangle = \langle v, f\rangle$ for all f in the convex hull of $\mathcal{E}(B_{X^*})$. Finally, the Krein-Milman theorem implies that the equality $\langle u, f\rangle = \langle v, f\rangle$ holds for all $f \in X^*$. This is a contradiction because continuous linear functionals separate points of X. □

Proof of Theorem 1.3. Assume that $(x_n)_{n\geq 1}$ converges weakly to x. Suppose now that there is a metric functional $\mathbf{h}^* \in X^{\diamondsuit}$ such that

$$\liminf_{n\to\infty} \mathbf{h}^*(x_n) < \mathbf{h}^*(x). \tag{4.2}$$

Clearly, $\mathbf{h}^*$ is not the trivial zero functional. Denote by A the liminf in the inequality (4.2). Since $\mathbf{h}^*$ is a metric funtional, we have $\mathbf{h}^*(x_n) \geq -\|x_n\|$ for all $n \geq 1$, hence A is finite. We pick a subsequence $(x_{n_i})_{i\geq 1}$ such that $\mathbf{h}^*(x_{n_i}) \to A$. Since $A < \mathbf{h}^*(x)$, there exists a positive real number ϵ such that $A + \epsilon < \mathbf{h}^*(x)$. Then, there exists a positive integer i_0 such that for every $i \geq i_0$ we have

$$\mathbf{h}^*(x_{n_i}) \leq A + \epsilon < \mathbf{h}^*(x). \tag{4.3}$$

Now, consider the set

$$Y = \{y \in X \mid \mathbf{h}^*(y) \leq A + \epsilon\}.$$

Since $\mathbf{h}^*$ is a metric functional, Y is a closed convex subset of X. From (4.3) it follows that Y is nonempty and $x \in X \setminus Y$. A classical separation theorem implies that there exists a continuous linear functional $f \in X^*$ and a real number B such that $\langle x_{n_i}, f\rangle \geq B > \langle x, f\rangle$ for all $i \geq i_0$. This contradicts the assumption of weak convergence. So, such a metric functional $\mathbf{h}^*$ that satisfies (4.2) cannot exists.

Now, we are going to prove the other direction of our claim. We assume that for every metric functional $\mathbf{h} \in X^{\diamondsuit}$ we have

$$\liminf_{n\to\infty} \mathbf{h}(x_n) \geq \mathbf{h}(x). \tag{4.4}$$

By a result of Walsh ([Wal18, Corollary 3.5]), extreme points of the closed unit ball B_{X^*} of X^* are elements in $X^{\diamondsuit}$. Thus, the inequality (4.4) holds for all extreme points $\pm a^*$ of B_{X^*}. Then, we have

$$\lim_{n\to\infty} \langle -x_n, a^* \rangle = \langle -x, a^* \rangle.$$

Since (x_n) is bounded, the above equality is equivalent to weak convergence due to Rainwater's theorem [Rai63]. □

Proof of Theorem 1.4. In ℓ_1, the claim follows from Proposition 3.12.

Consider now the space $C[0,1]$ equipped with the sup-norm. For each t in $[0,1]$, both functionals $f \mapsto f(t)$ and $f \mapsto -f(t)$ are metric functionals on $C[0,1]$. This follows from the explicit formulas given in [Wal18, Theorem 5.1]. If a sequence $(f_n)_{n\geq 1}$ in $C[0,1]$ is unbounded, then there is a real number τ in $[0,1]$ and a sequence of positive integers $(n_i)_{i\geq 1}$ such that

$$|f_{n_i}(\tau)| \to \infty.$$

If we have $f_{n_i}(\tau) \to -\infty$, use the metric functional $\mathbf{h}(f) = f(\tau)$ to get $\mathbf{h}(f_{n_i}) \to -\infty$. Thus, the sequence $(f_n)_{n\geq 1}$ cannot converge $d-$weakly. If we have $f_{n_i}(\tau) \to +\infty$, use the metric functional $\mathbf{h}(f) = -f(\tau)$ instead.

Finally, assume that X is a normed linear space whose dual X^* is strictly convex. We know that extreme points of the closed unit ball B_{X^*} are metric functionals [Wal18, Corollary 3.5]. It is also known that, in the present case, all continuous linear functionals of norm 1 are extreme points of B_{X^*}. Thus, every sequence in X that converges $d-$weakly, also converges in the standard weak topology, and hence must be bounded. □

To prove Theorem 1.6 we need the following result.

Lemma 4.1. *Let $(X, \|\cdot\|)$ be a normed linear space. Suppose that s and t are real numbers and v is a vector in X. If $\mathbf{h}$ is a metric functional on X, then there exists a metric functional $\boldsymbol{\eta}$ on X such that*

$$\mathbf{h}(sx + tv) = |s|\boldsymbol{\eta}(x) + \mathbf{h}(tv),$$

for all x in X.

Proof. The claim trivially holds for the case $s = 0$. Now, suppose that s is a nonzero real number. If $\mathbf{h}$ is a metric functional on X, then there is a net (w_α) in X such that

$$\mathbf{h}(\cdot) = \lim_{\alpha} (\|\cdot - w_\alpha\| - \|w_\alpha\|).$$

Consider the net (z_α) in X with $z_\alpha = s^{-1}(w_\alpha - tv)$. Then, for every x in X we have

$$\|sx + tv - w_\alpha\| - \|w_\alpha\| = |s|(\|x - z_\alpha\| - \|z_\alpha\|) + (\|tv - w_\alpha\| - \|w_\alpha\|).$$

Thus, our claim holds with

$$\boldsymbol{\eta}(x) = \lim_{\alpha}(\|x - z_\alpha\| - \|z_\alpha\|).$$

□

Proof of Theorem 1.6. Let $\mathbf{h}$ be an arbitrary metric functional on X. By Lemma 4.1, there exists a metric functional $\boldsymbol{\eta}$ on X such that

$$\mathbf{h}(sx + tv) = |s|\boldsymbol{\eta}(x) + \mathbf{h}(tv),$$

for all x in X. Thus, the hypothesis $x_n \xrightarrow{d} u$ implies

$$\begin{aligned}\liminf_{n\to\infty} \mathbf{h}(sx_n + tv) &= |s| \liminf_{n\to\infty} \boldsymbol{\eta}(x_n) + \mathbf{h}(tv) \\ &\geq |s|\boldsymbol{\eta}(u) + \mathbf{h}(tv) \\ &= \mathbf{h}(su + tv).\end{aligned}$$

Now, since $\mathbf{h}$ is a metric functional and the hypothesis $d(y_n, v) \to 0$ holds, we have

$$\liminf_{n\to\infty} \mathbf{h}(sx_n + ty_n) = \liminf_{n\to\infty} \mathbf{h}(sx_n + tv).$$

Therefore, we have $sx_n + ty_n \xrightarrow{d} su + tv$. □

Proof of Theorem 1.7. Denote by $\Lambda_d(x_n)$ the set of all points z such that $x_n \xrightarrow{d} z$. We prove first that the set $\Lambda_d(x_n)$ is convex. Let z_1 and z_2 be two points in $\Lambda_d(x_n)$ and s be a real number in the interval $[0, 1]$. Let $\mathbf{h}$ be a metric functional on the $W-$convex metric space (X, d). By Proposition 2.8, the metric functional $\mathbf{h}$ is $W-$convex. Thus, we have

$$\begin{aligned}\mathbf{h}(W(z_1, z_2, s)) &\leq (1 - s)\,\mathbf{h}(z_1) + s\,\mathbf{h}(z_2) \\ &\leq (1 - s) \liminf_{n\to\infty} \mathbf{h}(x_n) + s \liminf_{n\to\infty} \mathbf{h}(x_n) \\ &\leq \liminf_{n\to\infty} \mathbf{h}(x_n).\end{aligned}$$

This shows that $x_n \xrightarrow{d} W(z_1, z_2, s)$ for all $s \in [0,1]$. Therefore, the set $\Lambda_d(x_n)$ is convex.

To show that the set $\Lambda_d(x_n)$ is closed we need to recall that every metric functional is $1-$Lipschitz. If $(z_m)_{m\geq 1}$ is a sequence of points in $\Lambda_d(x_n)$, then for every metric functional $\mathbf{h}$ and every $m \geq 1$ we have

$$\liminf_{n\to\infty} \mathbf{h}(x_n) \geq \mathbf{h}(z_m).$$

So, if we have $d(z_m, z) \to 0$, then z is in $\Lambda_d(x_n)$. □

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