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On the Metric Compactification of Infinite-dimensional ℓ_p Spaces

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Abstract. The notion of metric compactification was introduced by Gromov and later rediscovered by Rieffel. It has been mainly studied on proper geodesic metric spaces. We present here a generalization of the metric compactification that can be applied to infinite-dimensional Banach spaces. Thereafter we give a complete description of the metric compactification of infinite-dimensional ℓ_p spaces for all $1 \leqslant p < \infty$. We also give a full characterization of the metric compactification of infinite-dimensional Hilbert spaces.

1 Introduction

In connection with topology and potential theory, Gromov [2,8] introduced a method for attaching an ideal boundary X_∞ at infinity of a metric space X. The method consists in mapping X into the set of real-valued continuous functions $C(X)$ equipped with the topology of uniform convergence on bounded subsets. If the metric space X is proper and geodesic, then Gromov's bordification $X \sqcup X_\infty$ becomes a compact topological space that contains X as a dense open subset [1,3]. Later, Rieffel [17] obtained the compact space $X \sqcup X_\infty$ as the maximal ideal space of a unital commutative C^*-algebra, and termed it the *metric compactification*, although it is perhaps more often known as the *horofunction compactification*. For finite-dimensional Banach spaces, this compactification has been studied in [5,9–11,18].

Compactness is a fundamental tool in mathematics. Infinite-dimensional Banach spaces are not locally compact, so Gromov's procedure does not yield a compactification, only a bordification. However, there is an alternative method to compactify an infinite-dimensional Banach space X. By considering instead the topology of pointwise convergence on $C(X)$ and following [6,7,16] we obtain a metric compactification of X in a weak sense; see Section 2. While studying certain metric geometries of infinite-dimensional convex cones, Walsh [19] presented a description of a certain subset of the metric compactification of infinite-dimensional Banach spaces, namely the set of Busemann points. A Busemann point is an element of the metric compactification obtained as a limit of some almost-geodesic net, a concept that was first introduced by Rieffel and then slightly modified by Walsh.

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We emphasize that the techniques we use in this paper are significantly different from those used by Walsh. Moreover, we provide explicit formulas for all the elements of the metric compactification of some classical infinite-dimensional Banach spaces.

Let J be any nonempty index set. We denote by $x = (x(j))_{j\in J}$ any element of the real vector space $\mathbb{R}^J$. For every $1 \le p < \infty$ we consider the p-norm $\|\cdot\|_p$ on $\mathbb{R}^J$ defined by $\|x\|_p := \left(\sum_{j\in J}|x(j)|^p\right)^{1/p}$, where the sum is given by

$$\sum_{j\in J}|x(j)|^p = \sup\left\{\sum_{j\in F}|x(j)|^p \;\middle|\; F \text{ a finite subset of } J\right\}, \tag{1.1}$$

for all $x \in \mathbb{R}^J$. We denote by $\ell_p(J)$ the space of all $x = (x(j))_{j\in J}$ in $\mathbb{R}^J$ such that $\|x\|_p$ is finite. That is, $\ell_p(J) = \left\{x \in \mathbb{R}^J \mid \|x\|_p < \infty\right\}$. The space $\ell_\infty(J)$ consists of all $x = (x(j))_{j\in J}$ in $\mathbb{R}^J$ such that $\sup_{j\in J}|x(j)|$ is finite.

In [9], the author gives a complete description of the metric compactification of $\ell_p(\{1,\dots,N\})$ for all $1 \le p \le \infty$ and $N \in \mathbb{N}$. In this paper, we study the metric compactification of $\ell_p(J)$ for all $1 \le p < \infty$ and J any countably infinite or uncountable index set.

The paper is organized as follows. In Section 2 we introduce the notion of metric compactification of infinite-dimensional Banach spaces. In Section 3 we give a complete description of the metric compactification of the infinite-dimensional Banach space $\ell_1(J)$. In Section 4 we give a complete description of the metric compactification of infinite-dimensional real Hilbert spaces. In Section 5 we give a complete description of the metric compactification of the infinite-dimensional Banach space $\ell_p(J)$ for all $1 < p < \infty$.

Recent works have shown that the metric compactification provides a powerful modern tool for the study of deterministic and random dynamics of nonexpansive mappings [6, 7, 12, 13]. The purpose of this paper is to present explicit formulas for all the elements of the metric compactification of infinite-dimensional ℓ_p spaces.

2 Preliminaries

2.1 The Metric Compactification

Let (X, d) be a metric space. Fix an arbitrary *base point* b in X. For each $y \in X$ we consider the element $h_{b,y}$ of $\mathbb{R}^X$ defined by

$$h_{b,y}(x) := d(x,y) - d(b,y), \text{ for all } x \in X. \tag{2.1}$$

For every $y \in X$, the function $h_{b,y}$ is bounded from below by $-d(b,y)$. Moreover, $\{h_{b,y} \mid y \in X\}$ is a family of 1-Lipschitz functions with respect to the metric d. Indeed, by the triangle inequality we have

$$\begin{aligned}|h_{b,y}(x) - h_{b,y}(z)| &= |d(x,y) - d(b,y) - d(z,y) + d(b,y)|\\ &= |d(x,y) - d(z,y)| \le d(x,z),\end{aligned}$$

for all $x, z \in X$. Furthermore, by taking $z = b$ we obtain $|h_{b,y}(x)| \leq d(x,b)$ for all $x \in X$. Hence,

$$\{h_{b,y} \mid y \in X\} \subset \prod_{x \in X} [-d(x,b), d(x,b)]. \tag{2.2}$$

By Tychonoff's theorem, the set on the right-hand side of (2.2) is compact in the product topology. Therefore, the set $\{h_{b,y} \mid y \in X\}$ has compact closure in this topology, which is equivalent to the topology of pointwise convergence. Moreover, for any two different base points $b, b' \in X$, the equality

$$h_{b,y}(x) - h_{b,y}(b') = h_{b',y}(x) \tag{2.3}$$

holds for all $x \in X$. Then (2.3) induces a homeomorphism between the closures $\mathrm{cl}(\{h_{b,y} \mid y \in X\})$ and $\mathrm{cl}(\{h_{b',y} \mid y \in X\})$. We write h_y instead of $h_{b,y}$, and so we say that

$$\overline{X}^h := \mathrm{cl}\left(\{h_y \mid y \in X\}\right) \tag{2.4}$$

is the *metric compactification* of (X, d). As in [7], the elements of $\overline{X}^h$ are called *metric functionals* on X. By following [16], it will be convenient to consider the partition $\overline{X}^h = \overline{X}^{h,F} \sqcup \overline{X}^{h,\infty}$, where

$$\overline{X}^{h,F} := \left\{ h \in \overline{X}^h \;\middle|\; \inf_{x \in X} h(x) > -\infty \right\},$$
$$\overline{X}^{h,\infty} := \left\{ h \in \overline{X}^h \;\middle|\; \inf_{x \in X} h(x) = -\infty \right\}.$$

The elements of $\overline{X}^{h,F}$ are called *finite metric functionals*, while the elements of $\overline{X}^{h,\infty}$ are called *metric functionals at infinity*. It is clear that $\overline{X}^{h,F}$ contains the set $\{h_y \mid y \in X\}$ of *internal metric functionals*.

Remark 2.1 Recall that a metric space (X, d) is geodesic if every pair of points $x, y \in X$ can be joined by a path $\gamma\colon [0, d(x,y)] \to X$ such that $\gamma(0) = x$, $\gamma(d(x,y)) = y$ and $d(\gamma(s), \gamma(t)) = |s - t|$ for all s, t. We say that X is proper if every closed ball of X is compact. If (X, d) is a proper geodesic metric space, then the set $\overline{X}^h$ in (2.4) coincides with the usual Gromov metric compactification. Indeed, since X is proper and $\{h_y \mid y \in X\}$ is a family of 1-Lipschitz functions, it follows by the Arzelà-Ascoli theorem that the topology of pointwise convergence and the topology of uniform convergence on bounded subsets will produce the same compact closure (2.4). Moreover, since X is geodesic, the set of internal metric functionals $\{h_y \mid y \in X\}$ can be identified with X, which becomes a dense open set in $\overline{X}^h$. The set

$$\partial_h X := \overline{X}^h \setminus X$$

is called the *horofunction boundary* of X. For general metric spaces, however, although the space $\overline{X}^h$ is always compact with respect to the pointwise topology, the continuous injection $y \mapsto h_y$ may not be an embedding from X to $\overline{X}^h$; see [6] for more details.

Remark 2.2 If X is a Banach space with metric induced by a norm $\|\cdot\|$, we choose the base point $b = 0 \in X$ so (2.1) becomes

$$h_y(x) = \|x - y\| - \|y\|.$$

If X is a finite-dimensional Banach space, then every metric functional $h \in \overline{X}^h$ can be written as $h = \lim_{n\to\infty} h_{y_n}$, for some sequence $\{y_n\}_{n\in\mathbb{N}}$ in X. For an infinite-dimensional Banach space X, the compact space $\overline{X}^h$ may not be metrizable. However, for every metric functional $h \in \overline{X}^h$ there will always exist a net $\{y_\alpha\}_\alpha$ in X such that $h_{y_\alpha} \underset{\alpha}{\longrightarrow} h$ pointwise on X. For details about convergence of nets, we refer the reader to [14].

The following is a characterization of the finite metric functionals on Banach spaces.

Lemma 2.3 *If $\{y_\alpha\}_\alpha$ is a bounded net in a Banach space $(X, \|\cdot\|)$ such that $h_{y_\alpha} \underset{\alpha}{\longrightarrow} h$ pointwise on X, then $h \in \overline{X}^{h,F}$.*

Proof Indeed, by passing to a subnet we may assume that $\|y_\alpha\| \underset{\alpha}{\longrightarrow} c$. Also, for every $x \in X$ and for every α, we have

$$h_{y_\alpha}(x) = \|x - y_\alpha\| - \|y_\alpha\| \geq -\|y_\alpha\|.$$

Then $h(x) \geq -c$ for all $x \in X$, so the claim follows. ■

It is important to notice that if X is a finite-dimensional Banach space, then $\overline{X}^{h,F} = \{h_y \mid y \in X\}$. For infinite-dimensional Banach spaces, the set $\overline{X}^{h,F}$ of finite metric functionals may be larger than the set $\{h_y \mid y \in X\}$ of internal metric functionals.

2.2 Some Facts in Banach Space Theory

For convenience of the reader we establish some notation and recall some important facts in Banach space theory that will be useful in the following sections. Let $(X, \|\cdot\|)$ be a Banach space with dual space $(X^*, \|\cdot\|_*)$. Let $\{x_\alpha\}_\alpha$ be any net in X, and let x be any vector in X. We say that x_α converges strongly to x, and denote it by $x_\alpha \underset{\alpha}{\longrightarrow} x$, if $\lim_\alpha \|x_\alpha - x\| = 0$. We say that x_α converges weakly to x, and denote it by $x_\alpha \overset{w}{\underset{\alpha}{\longrightarrow}} x$, if $\lim_\alpha v(x_\alpha) = v(x)$ for all $v \in X^*$. Let $\{\mu_\alpha\}_\alpha$ be any net in X^* and let μ be any vector in X^*. We say that μ_α converges to μ in the weak-star topology, and denote it by $\mu_\alpha \overset{w^*}{\underset{\alpha}{\longrightarrow}} \mu$, if $\lim_\alpha \mu_\alpha(x) = \mu(x)$ for all $x \in X$.

Throughout we will make use of Alaoglu's theorem [15, Prop. 6.13], which states that the closed unit ball of the dual space X^* is compact in the weak-star topology. Likewise, it is well known that a Banach space X is reflexive if and only if its closed unit ball is compact in the weak topology; see [15, Prop. 6.14]. In particular, every bounded net in a reflexive Banach space has a weakly convergent subnet.

In the following sections we will denote by $c_0(J)$ the space of all $x \in \ell_\infty(J)$ such that the set $\{j \in J \mid |x(j)| \geq \epsilon\}$ is finite for all $\epsilon > 0$. The space $c_0(J)$ is a Banach space with respect to the norm inherited from $\ell_\infty(J)$. We denote by $c_{00}(J)$ the space of all $x \in \ell_\infty(J)$ such that the set $\{j \in J \mid x(j) \neq 0\}$ is finite. For every $x \in \ell_p(J)$ the sum in (1.1) is finite, so by considering the set inclusion $\subseteq$ as a partial order on the set of all finite subsets F of J, the set $\left\{\sum_{j\in F} |x(j)|^p\right\}_F$ is a net in $\mathbb{R}$ that converges to

$\sum_{j\in J}|x(j)|^p = \|x\|_p^p$ for all $1 \le p < \infty$. Therefore, it follows that $c_{00}(J)$ is dense in $\ell_p(J)$ for all $1 \le p < \infty$.

3 The Metric Compactification of ℓ_1

Throughout this section, for each $y \in \ell_1(J)$ we denote by h_y the function defined on $\ell_1(J)$ by

$$h_y(x) := \|x - y\|_1 - \|y\|_1, \text{ for all } x \in \ell_1(J).$$

In order to find possible metric functionals on $\ell_1(J)$ we will need the following argument, which was also used in [9, Lemma 3.1] to describe the metric compactification of the finite-dimensional space $\ell_1(\{1,\dots,N\})$.

Proposition 3.1 *Let $\{a_\beta\}_\beta$ be a net of real numbers. Then for every $r \in \mathbb{R}$,*

$$|r - a_\beta| - |a_\beta| \underset{\beta}{\longrightarrow} \begin{cases} -r & \text{if } a_\beta \underset{\beta}{\longrightarrow} +\infty, \\ r & \text{if } a_\beta \underset{\beta}{\longrightarrow} -\infty. \end{cases}$$

Moreover, since $\{|\cdot - a_\beta| - |a_\beta|\}_\beta$ is a family of 1-Lipschitz functions on $\mathbb{R}$, the convergence above is uniform on compact subsets of $\mathbb{R}$.

Remark 3.2 For every $x \in c_{00}(J)$ there exists a finite subset F of J such that $x(j) \neq 0$ for all $j \in F$, and $x(j) = 0$ for all $j \in J \setminus F$. Therefore,

$$\begin{aligned} h_y(x) &= \sum_{j\in J}|x(j) - y(j)| - \sum_{j\in J}|y(j)| \\ &= \sum_{j\in F}\big(|x(j) - y(j)| - |y(j)|\big). \end{aligned}$$

Lemma 3.3 *Let $\{y_\alpha\}_\alpha$ be any net in $\ell_1(J)$. Then there exists a subnet $\{y_\beta\}_\beta$, a subset I of J, a vector of signs $\varepsilon \in \{-1,+1\}^I$, and a vector $z \in \mathbb{R}^{J\setminus I}$ such that the net $\{h_{y_\beta}\}_\beta$ converges pointwise on $\ell_1(J)$ to the function*

$$x \longmapsto h^{\{I,\varepsilon,z\}}(x) := \sum_{j\in I}\varepsilon(j)x(j) + \sum_{j\in J\setminus I}\big(|x(j) - z(j)| - |z(j)|\big).$$

Proof Since $\ell_1(J) \subset \mathbb{R}^J \subset [-\infty,+\infty]^J$, we can think of $\{y_\alpha\}_\alpha$ as a net in the compact topological space $[-\infty,+\infty]^J$ with respect to the product topology. Therefore, there exists a subnet $\{y_\beta\}_\beta$ and a vector $\widetilde{z} \in [-\infty,+\infty]^J$ such that

$$y_\beta(j) \underset{\beta}{\longrightarrow} \widetilde{z}(j), \text{ for all } j \in J.$$

Hence, there exists a subset I (possibly empty) of J such that $\widetilde{z}(j) \in \{-\infty,+\infty\}$ for all $j \in I$, and $\widetilde{z}(j) \in \mathbb{R}$ for all $j \in J \setminus I$. Put $z(j) = \widetilde{z}(j)$ for all $j \in J \setminus I$, and for every $j \in I$,

$$\varepsilon(j) = \begin{cases} -1 & \text{if } \widetilde{z}(j) = +\infty \\ +1 & \text{if } \widetilde{z}(j) = -\infty. \end{cases}$$

Now let x be any element of $c_{00}(J)$. It follows from Remark 3.2 that there exists a finite subset F of J such that $x(j) \neq 0$ for all $j \in F$, and $x(j) = 0$ for all $j \in J \setminus F$ for which

$$h_{y_\beta}(x) = \sum_{j \in F \cap I} \left(|x(j) - y_\beta(j)| - |y_\beta(j)| \right) + \sum_{j \in F \cap (J \setminus I)} \left(|x(j) - y_\beta(j)| - |y_\beta(j)| \right).$$

By Proposition 3.1, we obtain

$$\begin{aligned} \lim_\beta h_{y_\beta}(x) &= \sum_{j \in F \cap I} \varepsilon(j) x(j) + \sum_{j \in F \cap (J \setminus I)} \left(|x(j) - z(j)| - |z(j)| \right) \\ &= \sum_{j \in I} \varepsilon(j) x(j) + \sum_{j \in J \setminus I} \left(|x(j) - z(j)| - |z(j)| \right). \end{aligned}$$

Therefore, the claim of the lemma follows readily from the fact that $c_{00}(J)$ is dense in $\ell_1(J)$ and $\{h_{y_\beta}\}_\beta$ is a family of 1-Lipschitz functions on $\ell_1(J)$. ■

Example 3.4 For simplicity let us assume that $J = \mathbb{N}$. Consider the following sequences in $\ell_1(\mathbb{N})$:

$$\begin{aligned} y_n &= (1, 0, \dots, 0, \underset{n^{\text{th}}}{n}, 0, 0, \dots), \\ \widetilde{y}_n &= (n, 1, 1, \dots, \underset{n^{\text{th}}}{1}, 0, 0, \dots). \end{aligned}$$

Then for every $x \in \ell_1(\mathbb{N})$, we have

$$\lim_{n \to \infty} h_{y_n}(x) = \left(|x(1) - 1| - 1 \right) + \sum_{j=2}^{\infty} |x(j)| = h_z(x), \tag{3.1}$$

$$\lim_{n \to \infty} h_{\widetilde{y}_n}(x) = -x(1) + \sum_{j=2}^{\infty} (|x(j) - 1| - 1) = h^{\{I, \varepsilon, z\}}(x), \tag{3.2}$$

where $z = (1, 0, 0, \dots) \in \ell_1(\mathbb{N})$ in (3.1), whereas $I = \{1\} \subset \mathbb{N}$, $\varepsilon(1) = -1$, $z(j) = 1$ for all $j \geq 2$ in (3.2). Notice that in both cases, we have $\|y_n\|_1 \to \infty$ and $\|\widetilde{y}_n\|_1 \to \infty$, as $n \to \infty$. However, $h^{\{I, \varepsilon, z\}}$ in (3.2) is a metric functional at infinity, while h_z in (3.1) is an internal metric functional.

The following result shows that bounded nets in $\ell_1(J)$ produce only internal metric functionals.

Lemma 3.5 *Let $\{y_\alpha\}_\alpha$ be a bounded net in $\ell_1(J)$. Then there exists a subnet $\{y_\beta\}_\beta$ and a vector $z \in \ell_1(J)$ such that the net $\{h_{y_\beta}\}_\beta$ converges pointwise to the function h_z.*

Proof Recall that $\ell_1(J)$ and $c_0(J)^*$ are isometrically isomorphic under the surjective isometry $L\colon \ell_1(J) \to c_0(J)^*$, $y \mapsto L_y$ defined by $L_y(x) = \sum_{j \in J} x(j) y(j)$, for all $x \in c_0(J)$. Therefore, we can consider the bounded net $\{y_\alpha\}_\alpha$ in $\ell_1(J)$ as a bounded net of continuous linear functionals on $c_0(J)$. Hence, by Alaoglu's theorem, there exists a subnet $\{y_\beta\}_\beta$ and a vector $z \in \ell_1(J)$ such that $y_\beta \xrightarrow[\beta]{w^*} z$. In particular, we have

$y_\beta(j) \underset{\beta}{\longrightarrow} z(j)$ for all $j \in J$. Now let x be any vector in $c_{00}(J)$. By Remark 3.2, there exists a finite subset F of J such that $x(j) \neq 0$ for all $j \in F$, and $x(j) = 0$ for all $j \in J \smallsetminus F$. Hence,

$$\begin{aligned} h_{y_\beta}(x) &= \sum_{j\in F} |x(j) - y_\beta(j)| - \sum_{j\in F} |y_\beta(j)| \\ &\underset{\beta}{\longrightarrow} \sum_{j\in F} |x(j) - z(j)| - \sum_{j\in F} |z(j)|. \end{aligned}$$

Therefore, since $z \in \ell_1(J)$ and $x(j) = 0$ for all $j \in J \smallsetminus F$, we obtain

$$\lim_\beta h_{y_\beta}(x) = \sum_{j\in J} |x(j) - z(j)| - \sum_{j\in J} |z(j)| = h_z(x).$$

Finally, since $c_{00}(J)$ is dense in $\ell_1(J)$ and $\{h_{y_\beta}\}_\beta$ is a family of 1-Lipschitz functions on $\ell_1(J)$, the claim follows. ∎

Theorem 3.6 *The metric compactification of $\ell_1(J)$ is given by*

$$\overline{\ell_1(J)}^h = \left\{ h^{\{I,\varepsilon,z\}} \in \mathbb{R}^{\ell_1(J)} \;\middle|\; \varnothing \subseteq I \subseteq J,\ \varepsilon \in \{-1,+1\}^I,\ z \in \mathbb{R}^{J\smallsetminus I} \right\}, \tag{3.3}$$

where for every $x \in \ell_1(J)$,

$$h^{\{I,\varepsilon,z\}}(x) = \sum_{j\in I} \varepsilon(j)x(j) + \sum_{j\in J\smallsetminus I} \big(|x(j) - z(j)| - |z(j)|\big).$$

Proof Let h be any element of $\overline{\ell_1(J)}^h$. Then there exists a net $\{y_\alpha\}_\alpha$ in $\ell_1(J)$ such that $h_{y_\alpha} \underset{\alpha}{\longrightarrow} h$ pointwise on $\ell_1(J)$. By Lemma 3.3, there exists a subnet $\{h_{y_\beta}\}_\beta$ of $\{h_{y_\alpha}\}_\alpha$, a subset I of J, a vector of signs $\varepsilon \in \{-1,+1\}^I$, and a vector $z \in \mathbb{R}^{J\smallsetminus I}$ such that for every $x \in \ell_1(J)$

$$\lim_\beta h_{y_\beta}(x) = h^{\{I,\varepsilon,z\}}(x) = \sum_{j\in I} \varepsilon(j)x(j) + \sum_{j\in J\smallsetminus I} \big(|x(j) - z(j)| - |z(j)|\big).$$

Then $h = h^{\{I,\varepsilon,z\}}$, and so h belongs to the set on the right-hand side of (3.3).

Suppose now that $h \in \mathbb{R}^{\ell_1(J)}$ is defined for every $x \in \ell_1(J)$ by

$$h(x) = \sum_{j\in I} \varepsilon(j)x(j) + \sum_{j\in J\smallsetminus I} \big(|x(j) - z(j)| - |z(j)|\big) \tag{3.4}$$

for some subset I of J, some $\varepsilon \in \{-1,+1\}^I$, and some $z \in \mathbb{R}^{J\smallsetminus I}$. Then for each finite subset F (with cardinality $\#F$) of J define

$$y_F(j) := \begin{cases} -\varepsilon(j)\#F & \text{if } j \in F \cap I, \\ z(j), & \text{if } j \in F \cap (J \smallsetminus I), \\ 0 & \text{if } j \in J \smallsetminus F. \end{cases}$$

With set inclusion $\subseteq$ as a partial order on the set of all finite subsets F of J, the set $\{y_F\}_F$ defines a net in $\ell_1(J)$ with norm

$$\|y_F\|_1 = \sum_{j\in J} |y_F(j)| = \sum_{j\in F\cap I} \#F + \sum_{j\in F\cap(J\smallsetminus I)} |z(j)|.$$

We claim that for every $x \in \ell_1(J)$ we have $h_{y_F}(x) \xrightarrow[F]{} h(x)$, where h is given in (3.4). If we prove this for all $x \in c_{00}(J)$, the claim follows readily from the fact that $c_{00}(J)$ is dense in $\ell_1(J)$ and the net $\{h_{y_F}\}_F$ is a family of 1-Lipschitz functions on $\ell_1(J)$. Let x be any element in $c_{00}(J)$. By Remark 3.2, there exists a finite set G of J such that

$$\begin{aligned} h_{y_F}(x) &= \sum_{j\in G}\big(|x(j)-y_F(j)|-|y_F(j)|\big) \\ &= \sum_{j\in G\cap F\cap I}\big(\big|-\varepsilon(j)x(j)-\#F\big|-\#F\big) \\ &\quad + \sum_{j\in G\cap F\cap(J\setminus I)}\big(|x(j)-z(j)|-|z(j)|\big) + \sum_{j\in G\cap(J\setminus F)}|x(j)|. \end{aligned} \tag{3.5}$$

As F gets larger, the third sum in (3.5) converges to 0, while the second sum converges to $\sum_{j\in G\cap(J\setminus I)}\big(|x(j)-z(j)|-|z(j)|\big)$. By Proposition 3.1, each term of the first sum in (3.5) converges to $\varepsilon(j)x(j)$, as F gets larger. Moreover, larger subsets F of J will eventually contain the finite set G. Hence, the first sum of (3.5) converges to $\sum_{j\in G\cap I}\varepsilon(j)x(j)$, as F gets larger. We have therefore shown that for every $x \in c_{00}(J)$,

$$h_{y_F}(x) \xrightarrow[F]{} \sum_{j\in G\cap I}\varepsilon(j)x(j) + \sum_{j\in G\cap(J\setminus I)}\big(|x(j)-z(j)|-|z(j)|\big) = h(x),$$

where G is the finite set $\{j \in J \mid x(j) \neq 0\}$. This concludes the proof of the theorem. ∎

By a simple inspection of all the elements of the set (3.3), we obtain the following corollary.

Corollary 3.7 *The set of finite metric functionals on $\ell_1(J)$ consists only of internal ones. That is, $\overline{\ell_1(J)}^{h,F} = \{h_y \mid y \in \ell_1(J)\}$.*

We will see in the next sections that Corollary 3.7 does not hold for infinite-dimensional Hilbert spaces and $\ell_p(J)$ with $1 < p < \infty$.

4 The Metric Compactification of Infinite-dimensional Hilbert Spaces

Throughout this section we will assume that $\mathcal{H}$ is an infinite-dimensional real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|x\| = \langle x,x\rangle^{1/2}$ for all $x \in \mathcal{H}$. For each $y \in \mathcal{H}$, we denote by h_y the function on $\mathcal{H}$ defined by

$$x \longmapsto h_y(x) := \|x-y\| - \|y\|.$$

Lemma 4.1 *Let $\{y_\alpha\}_\alpha$ be a bounded net in $\mathcal{H}$. Then there exists a subnet $\{y_\beta\}_\beta$, a vector $z \in \mathcal{H}$, and a real number $c \geq \|z\|$ such that the net $\{h_{y_\beta}\}_\beta$ converges pointwise on $\mathcal{H}$ to the function*

$$x \longmapsto h^{\{z,c\}}(x) := \big(\|x\|^2 - 2\langle x,z\rangle + c^2\big)^{1/2} - c.$$

Proof Let $\{y_\alpha\}_\alpha$ be a bounded net in $\mathcal{H}$. Since $\mathcal{H}$ is reflexive, by Alaoglu's theorem there exists a subnet $\{y_\beta\}_\beta$ and a vector $z \in \mathcal{H}$ such that

$$y_\beta \xrightarrow[\beta]{w} z. \tag{4.1}$$

By letting $c := \liminf_\beta \|y_\beta\|$, we have $\|z\| \le c$. Now, since $\{\|y_\beta\|\}_\beta$ is a bounded net in $\mathbb{R}$, by passing to a subnet, we can assume that

$$\lim_\beta \|y_\beta\| = c. \tag{4.2}$$

Let x be any element in $\mathcal{H}$. Then, by (4.1) and (4.2), we obtain

$$\begin{aligned} h_{y_\beta}(x) &= \|x - y_\beta\| - \|y_\beta\| \\ &= \left(\|x\|^2 - 2\langle x, y_\beta\rangle + \|y_\beta\|^2\right)^{1/2} - \|y_\beta\| \\ &\xrightarrow[\beta]{} \left(\|x\|^2 - 2\langle x, z\rangle + c^2\right)^{1/2} - c = h^{\{z,c\}}(x). \end{aligned}$$ ∎

Remark 4.2 (Radon–Riesz property) The weak limit (4.1) and the limit (4.2) of the norms $\|y_\beta\|$ in Lemma 4.1 imply

$$\begin{aligned} \|y_\beta - z\|^2 &= \|y_\beta\|^2 - 2\langle y_\beta, z\rangle + \|z\|^2 \\ &\xrightarrow[\beta]{} c^2 - 2\langle z, z\rangle + \|z\|^2 \\ &= c^2 - \|z\|^2. \end{aligned}$$

Hence y_β converges strongly to z if and only if $c = \|z\|$. It is clear that $h^{\{z,\|z\|\}} = h_z$.

Lemma 4.3 *Let $\{y_\alpha\}_\alpha$ be a net in $\mathcal{H}$ such that $\|y_\alpha\| \xrightarrow[\alpha]{} \infty$. Then there exists a subnet $\{y_\beta\}_\beta$ and a vector $z \in \mathcal{H}$ with $\|z\| \le 1$ such that the net $\{h_{y_\beta}\}_\beta$ converges pointwise on $\mathcal{H}$ to the function*

$$x \longmapsto h^{\{z\}}(x) := -\langle x, z\rangle.$$

Proof Since $\|y_\alpha\| \xrightarrow[\alpha]{} \infty$, by passing to a subnet, we can assume that $\|y_\alpha\| > 0$ for all α. Define $z_\alpha := y_\alpha/\|y_\alpha\|$ for all α. Since $\mathcal{H}$ is reflexive, it follows from Alaoglu's theorem that there exists a subnet $\{z_\beta\}_\beta$ and a vector $z \in \mathcal{H}$ with $\|z\| \le 1$ such that $z_\beta \xrightarrow[\beta]{w} z$. Let x be any vector in $\mathcal{H}$. Then

$$\begin{aligned} \|x - y_\beta\| &= \left(\|x\|^2 - 2\langle x, y_\beta\rangle + \|y_\beta\|^2\right)^{1/2} \\ &= \|y_\beta\|\left(\frac{\|x\|^2}{\|y_\beta\|^2} - \frac{2\langle x, z_\beta\rangle}{\|y_\beta\|} + 1\right)^{1/2}. \end{aligned} \tag{4.3}$$

It is clear that $\|y_\beta\| \xrightarrow[\beta]{} \infty$ implies

$$\frac{\|x\|^2}{\|y_\beta\|^2} - \frac{2\langle x, z_\beta\rangle}{\|y_\beta\|} \xrightarrow[\beta]{} 0. \tag{4.4}$$

Recall the Taylor expansion $\sqrt{a+1} = 1 + a/2 + O(a^2)$, when a is small enough. Hence by (4.4), we can write (4.3) as follows:

$$\begin{aligned} \|x - y_\beta\| &= \|y_\beta\| \left(1 + \frac{1}{2}\Big(\frac{\|x\|^2}{\|y_\beta\|^2} - \frac{2\langle x, z_\beta\rangle}{\|y_\beta\|} \Big) + O\Big(\frac{1}{\|y_\beta\|^2} \Big) \right) \\ &= \|y_\beta\| + \frac{1}{2}\Big(\frac{\|x\|^2}{\|y_\beta\|} - 2\langle x, z_\beta\rangle \Big) + O\Big(\frac{1}{\|y_\beta\|} \Big), \end{aligned}$$

as $\|y_\beta\| \xrightarrow[\beta]{} \infty$. Therefore, we obtain

$$\begin{aligned} \lim_\beta h_{y_\beta}(x) &= \lim_\beta \big[\, \|x - y_\beta\| - \|y_\beta\| \,\big] \\ &= \lim_\beta \left[\frac{1}{2}\Big(\frac{\|x\|^2}{\|y_\beta\|} - 2\langle x, z_\beta\rangle \Big) + O\Big(\frac{1}{\|y_\beta\|} \Big) \right] \\ &= -\langle x, z\rangle = h^{\{z\}}(x). \end{aligned}$$

■

Theorem 4.4 *The metric compactification $\overline{\mathcal{H}}^h = \overline{\mathcal{H}}^{h,F} \sqcup \overline{\mathcal{H}}^{h,\infty}$ of an infinite-dimensional real Hilbert space $\mathcal{H}$ is given by*

$$\begin{aligned} \overline{\mathcal{H}}^{h,F} &= \left\{ h^{\{z,c\}} \in \mathbb{R}^{\mathcal{H}} \;\middle|\; z \in \mathcal{H},\ c \geq \|z\| \right\} \cup \{0\}, \\ \overline{\mathcal{H}}^{h,\infty} &= \left\{ h^{\{z\}} \in \mathbb{R}^{\mathcal{H}} \;\middle|\; z \in \mathcal{H},\ 0 < \|z\| \leq 1 \right\}, \end{aligned}$$

where for every $x \in \mathcal{H}$,

$$\begin{aligned} h^{\{z,c\}}(x) &= \left(\|x\|^2 - 2\langle x, z\rangle + c^2 \right)^{1/2} - c, \\ h^{\{z\}}(x) &= -\langle x, z\rangle. \end{aligned}$$

Proof Let h be any element of $\overline{\mathcal{H}}^h$. Then there exists a net $\{y_\alpha\}_\alpha$ in $\mathcal{H}$ such that $h_{y_\alpha} \xrightarrow[\alpha]{} h$ pointwise on $\mathcal{H}$. Suppose that the net $\{y_\alpha\}_\alpha$ is bounded in $\mathcal{H}$. Then it follows from Lemma 4.1 that there exists a subnet $\{y_\beta\}_\beta$, a vector $z \in \mathcal{H}$, and a real number $c \geq \|z\|$ such that

$$\lim_\beta h_{y_\beta}(x) = \left(\|x\|^2 - 2\langle x, z\rangle + c^2 \right)^{1/2} - c = h^{\{z,c\}}(x),$$

for all $x \in \mathcal{H}$. Therefore, we obtain $h = h^{\{z,c\}}$, and hence $h \in \overline{\mathcal{H}}^{h,F} \setminus \{0\}$. Suppose now that the net $\{y_\alpha\}_\alpha$ is unbounded in $\mathcal{H}$. Then, by passing to a subnet we can assume that $\|y_\alpha\| \xrightarrow[\alpha]{} \infty$. It follows from Lemma 4.3 that there exists a subnet $\{y_\beta\}_\beta$ and a vector $z \in \mathcal{H}$ with $\|z\| \leq 1$ such that

$$\lim_\beta h_{y_\beta}(x) = -\langle x, z\rangle = h^{\{z\}}(x),$$

for all $x \in \mathcal{H}$. Hence, $h = h^{\{z\}}$ and so $h \in \overline{\mathcal{H}}^{h,\infty} \cup \{0\}$. Consequently, we have proved that the inclusion $\overline{\mathcal{H}}^h \subseteq \overline{\mathcal{H}}^{h,F} \sqcup \overline{\mathcal{H}}^{h,\infty}$ holds.

On the other hand, since $\mathcal{H}$ is infinite-dimensional and reflexive, there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$ with $\|u_n\| = 1$ for all $n \in \mathbb{N}$ such that $u_n \xrightarrow{w} 0$, as $n \to \infty$.

Suppose that $h^{\{z,c\}} \in \overline{\mathcal{H}}^{h,F} \setminus \{0\}$ for some $z \in \mathcal{H}$ and some $c \geq \|z\|$. Then for each $n \in \mathbb{N}$ define $y_n \in \mathcal{H}$ by

$$y_n := \left(c^2 - \|z\|^2\right)^{1/2} u_n + z.$$

It follows that $y_n \xrightarrow{w} z$, and also

$$\|y_n\|^2 = c^2 + 2\left(c^2 - \|z\|^2\right)^{1/2} \langle z, u_n \rangle \longrightarrow c^2,$$

as $n \to \infty$. Therefore, for every $x \in \mathcal{H}$,

$$\begin{aligned} h_{y_n}(x) = \|x - y_n\| - \|y_n\| &= \left(\|x\|^2 - 2\langle x, y_n \rangle + \|y_n\|^2\right)^{1/2} - \|y_n\| \\ &\xrightarrow[n\to\infty]{} \left(\|x\|^2 - 2\langle x, z \rangle + c^2\right)^{1/2} - c = h^{\{z,c\}}(x). \end{aligned}$$

Hence, $h^{\{z,c\}}$ is an element of $\overline{\mathcal{H}}^h$. Now suppose that $h \in \overline{\mathcal{H}}^{h,\infty} \cup \{0\}$. Then for every $x \in \mathcal{H}$, we have $h(x) = -\langle x, z \rangle$ for some $z \in \mathcal{H}$ with $\|z\| \leq 1$. For each $n \in \mathbb{N}$, define $\widetilde{y}_n \in \mathcal{H}$ by

$$\widetilde{y}_n := n\left(1 - \|z\|^2\right)^{1/2} u_n + nz.$$

Then

$$\|\widetilde{y}_n\|^2 = n^2\left(1 + 2\left(1 - \|z\|^2\right)^{1/2} \langle z, u_n \rangle\right) \longrightarrow \infty, \quad \text{as } n \to \infty.$$

Furthermore, we have $\widetilde{y}_n / \|\widetilde{y}_n\| \xrightarrow{w} z$, as $n \to \infty$. Then, by proceeding as in (4.3) and (4.4), we obtain

$$h_{\widetilde{y}_n}(x) \xrightarrow[n\to\infty]{} -\langle x, z \rangle = h(x),$$

for all $x \in \mathcal{H}$, and hence h is an element of $\overline{\mathcal{H}}^h$. We have therefore proved that the inclusion $\overline{\mathcal{H}}^{h,F} \sqcup \overline{\mathcal{H}}^{h,\infty} \subseteq \overline{\mathcal{H}}^h$ also holds. ∎

Remark 4.5 It readily follows from Remark 4.2 that $h^{\{z,c\}}$ is an internal metric functional if and only if $c = \|z\|$. Theorem 4.4 states that metric functionals on an infinite-dimensional real Hilbert space $\mathcal{H}$ are of three types:

(i) The set $\{h_y \mid y \in \mathcal{H}\}$ contains metric functionals that correspond to strongly convergent nets in $\mathcal{H}$.
(ii) The set $\{h^{\{z,c\}} \in \mathbb{R}^{\mathcal{H}} \mid z \in \mathcal{H},\ c > \|z\|\}$ contains *exotic* metric functionals that correspond to bounded nets converging weakly but not strongly in $\mathcal{H}$.
(iii) The set $\overline{\mathcal{H}}^{h,\infty} \cup \{0\}$ contains metric functionals that correspond to nets in $\mathcal{H}$ with norm tending to infinity.

Remark 4.6 For the infinite-dimensional real Hilbert space $\mathcal{H} = \ell_2(J)$ with norm $\|\cdot\|_2$, it follows readily from Theorem 4.4 that the metric compactification $\overline{\ell_2(J)}^h = \overline{\ell_2}^{h,F} \sqcup \overline{\ell_2}^{h,\infty}$ is given by

$$\begin{aligned} \overline{\ell_2}^{h,F} &= \left\{h^{\{z,c\}} \in \mathbb{R}^{\ell_2(J)} \mid z \in \ell_2(J),\ c \geq \|z\|_2\right\} \cup \{0\}, \\ \overline{\ell_2}^{h,\infty} &= \left\{h^{\{z\}} \in \mathbb{R}^{\ell_2(J)} \mid z \in \ell_2(J),\ 0 < \|z\|_2 \leq 1\right\}, \end{aligned}$$

where for every $x \in \ell_2(J)$,

$$h^{\{z,c\}}(x) = \left(\|x - z\|_2^2 + c^2 - \|z\|_2^2 \right)^{1/2} - c,$$
$$h^{\{z\}}(x) = -\sum_{j\in J} x(j)z(j).$$

This observation will be useful in Section 5 to identify all the metric functionals on the remaining ℓ_p spaces.

5 The Metric Compactification of ℓ_p, with $1 < p < \infty$

Throughout this section we assume $1 < p < \infty$. For each $y \in \ell_p(J)$, we denote by h_y the function on $\ell_p(J)$ given by

$$x \longmapsto h_y(x) := \|x - y\|_p - \|y\|_p.$$

Lemma 5.1 *Let $\{y_\alpha\}_\alpha$ be a bounded net in $\ell_p(J)$. Then there exists a subnet $\{y_\beta\}_\beta$, a vector $z \in \ell_p(J)$, and a real number $c \geq \|z\|_p$ such that the net $\{h_{y_\beta}\}_\beta$ converges pointwise on $\ell_p(J)$ to the function*

$$x \longmapsto h^{\{z,c\}}(x) := \left(\|x - z\|_p^p + c^p - \|z\|_p^p \right)^{1/p} - c.$$

Proof Since $\{y_\alpha\}_\alpha$ is a bounded net in $\ell_p(J)$ and $\ell_p(J)$ is reflexive, it follows from Alaoglu's theorem that there exists a subnet $\{y_\beta\}_\beta$, and a vector $z \in \ell_p(J)$ such that $y_\beta \xrightarrow[\beta]{w} z$. In particular, we have

$$y_\beta(j) \underset{\beta}{\longrightarrow} z(j), \quad \text{for all } j \in J. \tag{5.1}$$

Letting $c := \liminf_\beta \|y_\beta\|_p$, we have $\|z\|_p \leq c$. Moreover, since $\{\|y_\beta\|_p\}_\beta$ is a bounded net in $\mathbb{R}$, by passing to a subnet, we can assume that

$$\lim_\beta \|y_\beta\|_p = c. \tag{5.2}$$

Let x be any vector in $c_{00}(J)$. Then there exists a finite subset F of J such that $x(j) \neq 0$ for all $j \in F$, and $x(j) = 0$ otherwise. By applying (5.1) and (5.2), we obtain

$$\begin{aligned}
\|x - y_\beta\|_p^p &= \sum_{j\in J} |x(j) - y_\beta(j)|^p \\
&= \sum_{j\in F} |x(j) - y_\beta(j)|^p + \sum_{j\in J\setminus F} |y_\beta(j)|^p \\
&= \sum_{j\in F} |x(j) - y_\beta(j)|^p + \sum_{j\in J} |y_\beta(j)|^p - \sum_{j\in F} |y_\beta(j)|^p \\
\underset{\beta}{\longrightarrow} &\sum_{j\in F} |x(j) - z(j)|^p + c^p - \sum_{j\in F} |z(j)|^p \\
&= \sum_{j\in J} |x(j) - z(j)|^p + c^p - \sum_{j\in J} |z(j)|^p \\
&= \|x - z\|_p^p + c^p - \|z\|_p^p.
\end{aligned}$$

Therefore, for every $x \in c_{00}(J)$, we have

$$h_{y_\beta}(x) = \|x - y_\beta\|_p - \|y_\beta\|_p = \left(\|x - y_\beta\|_p^p\right)^{1/p} - \|y_\beta\|_p$$
$$\underset{\beta}{\longrightarrow} \left(\|x - z\|_p^p + c^p - \|z\|_p^p\right)^{1/p} - c = h^{\{z,c\}}(x).$$

Since $c_{00}(J)$ is dense in $\ell_p(J)$, and the set $\{h_{y_\beta}\}_\beta$ is a family of 1-Lipschitz functions on $\ell_p(J)$, it follows that $h_{y_\beta} \underset{\beta}{\longrightarrow} h^{\{z,c\}}$ pointwise on $\ell_p(J)$. ∎

Lemma 5.1 describes metric functionals on $\ell_p(J)$ that correspond to bounded nets in $\ell_p(J)$. We now characterize possible metric functionals on $\ell_p(J)$ that correspond to nets with p-norm tending to infinity. In fact, we go further and give a characterization of metric functionals that are obtained from nets with norm tending to infinity in any Banach space with uniformly convex dual. Afterwards, we turn to the specific case of $\ell_p(J)$, where we give a full characterization of its metric compactification.

A Banach space $(V, \|\cdot\|)$ is called *uniformly convex* if for every $0 < \epsilon \le 2$ there exists $\delta(\epsilon) > 0$ such that $\|u + v\| \le 2(1 - \delta)$ whenever $u, v \in V$ with $\|u\| = \|v\| = 1$ and $\|u - v\| \ge \epsilon$. A well-known result due to Clarkson [4] is that L_p and ℓ_p spaces are uniformly convex for all $1 < p < \infty$.

Remark 5.2 If $\{u_\beta\}_\beta$ and $\{v_\beta\}_\beta$ are nets in the unit sphere of a uniformly convex Banach space $(V, \|\cdot\|)$ such that $\|u_\beta + v_\beta\| \underset{\beta}{\longrightarrow} 2$, then we have $\|u_\beta - v_\beta\| \underset{\beta}{\longrightarrow} 0$. Indeed, if we suppose that $\|u_\beta - v_\beta\| \underset{\beta}{\not\to} 0$, then by passing to a subnet, we can assume that $\|u_\beta - v_\beta\| \ge \epsilon$ for some $\epsilon > 0$ and all β. By uniform convexity, there exists $\delta(\epsilon) > 0$ such that $\|u_\beta + v_\beta\| \le 2(1 - \delta)$, for all β. Therefore,

$$2 = \lim_\beta \|u_\beta + v_\beta\| \le 2 - 2\delta,$$

which is a contradiction.

Lemma 5.3 *Let $(X, \|\cdot\|)$ be an infinite-dimensional Banach space such that its dual space $(X^*, \|\cdot\|_*)$ is uniformly convex. Let $\{y_\alpha\}_\alpha$ be a net in X with $\|y_\alpha\| \underset{\alpha}{\longrightarrow} \infty$. Then there exists a subnet $\{y_\beta\}_\beta$ and a continuous linear functional $\mu \in X^*$ with $\|\mu\|_* \le 1$ such that $h_{y_\beta} \underset{\beta}{\longrightarrow} -\mu$ pointwise on X.*

Proof Without loss of generality, we can assume that $\|y_\alpha\| \ne 0$ for all α. Consider the net of unit vectors $\{z_\alpha\}_\alpha$ in X given by $z_\alpha = y_\alpha / \|y_\alpha\|$, for all α. By the Hahn–Banach theorem, for each α there exists $\mu_\alpha \in X^*$ with $\|\mu_\alpha\|_* = 1$ such that $\mu_\alpha(z_\alpha) = 1$. Hence, by Alaoglu's theorem, there exists a subnet $\{\mu_\beta\}_\beta$ and a continuous linear functional $\mu \in X^*$ with $\|\mu\|_* \le 1$ such that $\mu_\beta \overset{w^*}{\underset{\beta}{\longrightarrow}} \mu$. Now let x be any vector in X. By extracting a further subnet if necessary, we can assume that $x - y_\beta \ne 0$ for all β. Define the net of unit vectors $\{z_\beta^x\}_\beta$ in X by

$$z_\beta^x := \frac{y_\beta - x}{\|x - y_\beta\|}.$$

By the Hahn–Banach theorem, it follows that for each β there exists $\mu_\beta^x \in X^*$ with $\|\mu_\beta^x\|_* = 1$ such that

$$1 = \mu_\beta^x(z_\beta^x) = \frac{\mu_\beta^x(y_\beta) - \mu_\beta^x(x)}{\|x - y_\beta\|} = \frac{\mu_\beta^x(z_\beta) - \mu_\beta^x\left(\frac{x}{\|y_\beta\|}\right)}{\left\|\frac{x}{\|y_\beta\|} - z_\beta\right\|}. \tag{5.3}$$

On the other hand, for every β we have

$$-\frac{\|x\|}{\|y_\beta\|} + 1 \le \left\|\frac{x}{\|y_\beta\|} - z_\beta\right\| \le \frac{\|x\|}{\|y_\beta\|} + 1, \tag{5.4}$$

and also

$$\left|\mu_\beta^x\left(\frac{x}{\|y_\beta\|}\right)\right| \le \frac{\|x\|}{\|y_\beta\|}. \tag{5.5}$$

Hence, by applying the assumption $\|y_\beta\| \underset{\beta}{\longrightarrow} \infty$ in (5.4) and (5.5), we obtain

$$\left\|\frac{x}{\|y_\beta\|} - z_\beta\right\| \underset{\beta}{\longrightarrow} 1, \tag{5.6}$$

$$\mu_\beta^x\left(\frac{x}{\|y_\beta\|}\right) \underset{\beta}{\longrightarrow} 0. \tag{5.7}$$

Therefore, by applying (5.6) and (5.7) in (5.3), it follows that

$$\mu_\beta^x(z_\beta) \underset{\beta}{\longrightarrow} 1, \tag{5.8}$$

and hence,

$$2 \ge \|\mu_\beta^x + \mu_\beta\|_* \ge |(\mu_\beta^x + \mu_\beta)(z_\beta)| = |\mu_\beta^x(z_\beta) + \mu_\beta(z_\beta)| \underset{\beta}{\longrightarrow} 2.$$

Since X^* is uniformly convex, it follows from Remark 5.2 that

$$\|\mu_\beta^x - \mu_\beta\|_* \underset{\beta}{\longrightarrow} 0.$$

Then we obtain

$$\mu_\beta^x \overset{w^*}{\underset{\beta}{\longrightarrow}} \mu. \tag{5.9}$$

Finally, it follows from (5.3) that

$$\|x - y_\beta\| = -\mu_\beta^x(x) + \mu_\beta^x(z_\beta)\|y_\beta\|,$$

and therefore, by applying (5.8) and (5.9), we obtain

$$\begin{aligned} h_{y_\beta}(x) &= \|x - y_\beta\| - \|y_\beta\| \\ &= -\mu_\beta^x(x) + \mu_\beta^x(z_\beta)\|y_\beta\| - \|y_\beta\| \\ &\underset{\beta}{\longrightarrow} -\mu(x). \end{aligned}$$

∎

We are now ready to give a full characterization of the metric compactification of $\ell_p(J)$ for all $1 < p < \infty$.

Theorem 5.4 *Let $1 < p < \infty$. The metric compactification $\overline{\ell_p(J)}^h = \overline{\ell_p(J)}^{h,F} \sqcup \overline{\ell_p(J)}^{h,\infty}$ of the infinite-dimensional $\ell_p(J)$ space is given by*

$$\begin{aligned}\overline{\ell_p(J)}^{h,F} &= \left\{ h^{\{z,c\}} \in \mathbb{R}^{\ell_p(J)} \;\middle|\; z \in \ell_p(J),\ c \geq \|z\|_p \right\} \cup \{0\},\\ \overline{\ell_p(J)}^{h,\infty} &= \left\{ h^{\{\mu\}} \in \mathbb{R}^{\ell_p(J)} \;\middle|\; \mu \in \ell_{p/(p-1)}(J),\ 0 < \|\mu\|_{p/(p-1)} \leq 1 \right\},\end{aligned}$$

where for every $x \in \ell_p(J)$,

$$\begin{aligned}h^{\{z,c\}}(x) &= \left(\|x - z\|_p^p + c^p - \|z\|_p^p \right)^{1/p} - c,\\ h^{\{\mu\}}(x) &= -\sum_{j \in J} \mu(j) x(j).\end{aligned}$$

Proof Let $h \in \overline{\ell_p(J)}^h$. Then there exists a net $\{y_\alpha\}_\alpha$ in $\ell_p(J)$ such that $h_{y_\alpha} \underset{\alpha}{\longrightarrow} h$ pointwise on $\ell_p(J)$. Suppose that the net $\{y_\alpha\}_\alpha$ is bounded in $\ell_p(J)$. Then it follows from Lemma 5.1 that there exists a subnet $\{y_\beta\}_\beta$, a vector $z \in \ell_p(J)$, and a real number $c \geq \|z\|_p$ such that

$$\lim_\beta h_{y_\beta}(x) = \left(\|x - z\|_p^p + c^p - \|z\|_p^p \right)^{1/p} - c = h^{\{z,c\}}(x), \quad \text{for all } x \in \ell_p(J).$$

Hence, $h = h^{\{z,c\}}$ and so $h \in \overline{\ell_p(J)}^{h,F} \setminus \{0\}$. Now suppose that the net $\{y_\alpha\}_\alpha$ is unbounded in $\ell_p(J)$. Then, by passing to a subnet, we can assume that $\|y_\alpha\|_p \underset{\alpha}{\longrightarrow} \infty$. Let $q = p/(p-1)$. Since the dual space of $\ell_p(J)$ is the uniformly convex space $\ell_q(J)$, it follows from Lemma 5.3 and ℓ_p/ℓ_q duality that there exists $\mu \in \ell_q(J)$ with $\|\mu\|_q \leq 1$ such that

$$\lim_\beta h_{y_\beta}(x) = -\sum_{j \in J} \mu(j) x(j),$$

for all $x \in \ell_p(J)$. Therefore h belongs to $\overline{\ell_p(J)}^{h,\infty} \cup \{0\}$. We have proved that the inclusion

$$\overline{\ell_p(J)}^h \subseteq \overline{\ell_p(J)}^{h,F} \sqcup \overline{\ell_p(J)}^{h,\infty}$$

holds.

On the other hand, suppose that $h^{\{z,c\}} \in \overline{\ell_p(J)}^{h,F} \setminus \{0\}$, for some $z \in \ell_p(J)$ and some $c \geq \|z\|_p$. Let $a_{z,c} = (c^p - \|z\|_p^p)^{1/p}$. Pick any countably infinite subset $K = \{j_m\}_{m=1}^\infty$ of J, and for each $m \in \mathbb{N}$, define

$$y_m(j) := \begin{cases} a_{z,c} + z(j) & \text{if } j = j_m,\\ z(j) & \text{if } j \neq j_m.\end{cases} \tag{5.10}$$

It is clear that $y_m \in \ell_p(J)$ for all $m \in \mathbb{N}$. Moreover, we have $y_m \xrightarrow{w} z$ and $\|y_m\|_p^p \longrightarrow c^p$, as $m \to \infty$. Then, for every $x \in \ell_p(J)$, we have

$$\begin{aligned}\|x - y_m\|_p^p &= \sum_{j \neq j_m} |x(j) - z(j)|^p + |x(j_m) - a_{z,c} - z(j_m)|^p \\ &= \|x - z\|_p^p - |x(j_m) - z(j_m)|^p + |x(j_m) - a_{z,c} - z(j_m)|^p \\ &\underset{m\to\infty}{\longrightarrow} \|x - z\|_p^p + c^p - \|z\|_p^p.\end{aligned}$$

Therefore, (5.10) defines a bounded sequence in $\ell_p(J)$ such that

$$\begin{aligned}\lim_{m\to\infty} h_{y_m}(x) &= \lim_{m\to\infty} \left[\|x - y_m\|_p - \|y_m\|_p \right] \\ &= \left(\|x - z\|_p^p + c^p - \|z\|_p^p \right)^{1/p} - c \\ &= h^{\{z,c\}}(x), \text{ for all } x \in \ell_p(J).\end{aligned}$$

Hence, $h^{\{z,c\}} \in \overline{\ell_p(J)}^h$. Now suppose that $h \in \overline{\ell_p(J)}^{h,\infty} \cup \{0\}$. Then for every $x \in \ell_p(J)$, we have

$$h(x) = -\sum_{j\in J} \mu(j)x(j)$$

for some $\mu \in \ell_q(J)$ with $\|\mu\|_q \leq 1$ and $q = p/(p-1)$. Pick any countably infinite subset $K = \{j_m\}_{m=1}^{\infty}$ of J, and define the sequence $\{\mu_m\}_{m\in\mathbb{N}}$ in $\ell_q(J)$ by

$$\mu_m(j) := \begin{cases} \left(1 - \|\mu\|_q^q + |\mu(j)|^q\right)^{1/q} & \text{if } j = j_m, \\ \mu(j) & \text{if } j \neq j_m. \end{cases}$$

Then $\|\mu_m\|_q = 1$ for all $m \in \mathbb{N}$, and also $\mu_m \xrightarrow{w} \mu$. By ℓ_p/ℓ_q duality, it follows that for each $m \in \mathbb{N}$ there exists $z_m \in \ell_p(J)$ with $\|z_m\|_p = 1$ such that $\sum_{j\in J} \mu_m(j)z_m(j) = 1$. Therefore, by letting $y_m = mz_m$ and by proceeding as in the proof of Lemma 5.3, we obtain $h_{y_m}(x) \to h(x)$ for all $x \in \ell_p(J)$, as $m \to \infty$. Hence, h belongs to $\overline{\ell_p(J)}^h$. Consequently, the inclusion

$$\overline{\ell_p(J)}^{h,F} \sqcup \overline{\ell_p(J)}^{h,\infty} \subseteq \overline{\ell_p(J)}^h$$

also holds. ∎

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