

*THE HOROFUNCTION BOUNDARY OF
FINITE-DIMENSIONAL ℓ_p SPACES*

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Abstract. We give a complete description of the horofunction boundary of finite-dimensional ℓ_p spaces for $1 \leq p \leq \infty$. We also study the variation norm on $\mathbb{R}^{\mathcal{N}}$, $\mathcal{N} = \{1, \dots, N\}$, and the corresponding horofunction boundary. As a consequence, we describe the horofunctions for Hilbert's projective metric on the interior of the standard cone $\mathbb{R}_+^{\mathcal{N}}$ of $\mathbb{R}^{\mathcal{N}}$.

1. Introduction. There has recently been growing interest in the horofunction boundary of metric spaces. It is a powerful tool in the study of self-mappings of convex cones [9, 11] and random walks on groups [12]. The horofunction boundary has been studied mainly in spaces of nonpositive curvature since the introduction of the notion by Gromov [10]. By applying methods of convex analysis, Walsh [18] describes the horofunctions of general finite-dimensional normed spaces. Afterwards, in [19] he gives a description of the horofunction boundary of Hilbert's projective metric on general finite-dimensional cones. In an earlier paper [13] polyhedral normed spaces and Hilbert's projective metric on simplicial cones were studied.

Let $1 \leq p \leq \infty$ and let $\mathcal{N} = \{1, \dots, N\}$ for any $N \in \mathbb{N}$. Throughout, we shall denote by $\ell_p(\mathcal{N}, \mathbb{R})$ the vector space $\mathbb{R}^{\mathcal{N}}$ endowed with the norm

$$\|x\|_p = \begin{cases} (\sum_{i \in \mathcal{N}} |x_i|^p)^{1/p}, & 1 \leq p < \infty, \\ \max_{i \in \mathcal{N}} |x_i|, & p = \infty, \end{cases}$$

for all $x = (x_i)_{i \in \mathcal{N}} \in \mathbb{R}^{\mathcal{N}}$. We shall also denote by $\ell_{\text{var}}(\mathcal{N}, \mathbb{R})$ the vector space $\mathbb{R}^{\mathcal{N}}$ endowed with the pseudo-norm

$$\|x\|_{\text{var}} = \max_{i \in \mathcal{N}} x_i - \min_{i \in \mathcal{N}} x_i.$$

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The purpose of this paper is to give an explicit and detailed description of the horofunction boundary of $\ell_p(\mathcal{N}, \mathbb{R})$ for all $1 \leq p \leq \infty$. We also give a complete description of the horofunction boundary of the pseudo-normed space $\ell_{\text{var}}(\mathcal{N}, \mathbb{R})$. As a consequence, we readily obtain the horofunctions for Hilbert's projective metric on the interior of the standard cone $\mathbb{R}_+^{\mathcal{N}}$ of $\mathbb{R}^{\mathcal{N}}$.

We would like to emphasize that the techniques we use in this paper are significantly different from those used by Walsh. Our results contain explicit formulas for the horofunctions. This paper is organized as follows. In Section 3 we give a complete description of the horofunctions on $\ell_1(\mathcal{N}, \mathbb{R})$. In Section 4 we show that if $1 < p < \infty$ then the horofunction boundary of $\ell_p(\mathcal{N}, \mathbb{R})$ is precisely the set of all norm one linear functionals on $\ell_p(\mathcal{N}, \mathbb{R})$. In Section 5 we give a complete description of the horofunctions on $\ell_\infty(\mathcal{N}, \mathbb{R})$. In Section 6 we give a complete description of the horofunction boundary of $\ell_{\text{var}}(\mathcal{N}, \mathbb{R})$, and consequently we obtain all the horofunctions for Hilbert's projective metric on the interior of the standard cone $\mathbb{R}_+^{\mathcal{N}}$ of $\mathbb{R}^{\mathcal{N}}$. As an application of the latter result, we give a new proof of Perron's theorem.

2. Preliminaries: The horofunction boundary of a metric space.

Let (X, d) be a metric space. Fix an arbitrary *base point* b in X . Define the mapping $\tau_d : X \rightarrow \mathbb{R}^X$ by associating to any $y \in X$ the function $\tau_d(y)$ given by

$$(2.1) \quad \tau_d(y)(x) := d(x, y) - d(b, y)$$

for all x in X . For each $y \in X$, the function $\tau_d(y)$ is bounded from below by $-d(b, y)$ and, moreover, is 1-Lipschitz with respect to the metric d . In fact, by the triangle inequality it follows that

$$\begin{aligned} |\tau_d(y)(x) - \tau_d(y)(z)| &= |d(x, y) - d(b, y) - d(z, y) + d(b, y)| \\ &= |d(x, y) - d(z, y)| \leq d(x, z) \end{aligned}$$

for all $x, z \in X$. Furthermore, by taking $z = b$ we get $|\tau_d(y)(x)| \leq d(x, b)$ for all $x \in X$. Hence

$$\tau_d(X) \subset \prod_{x \in X} [-d(x, b), d(x, b)] \subset \mathbb{R}^X.$$

By Tychonoff's theorem the product space $\prod_{x \in X} [-d(x, b), d(x, b)]$ is compact in the product topology. Therefore the set $\tau_d(X)$ has compact closure in this topology, which is equivalent to the topology of pointwise convergence.

DEFINITION 2.1. We denote by $\overline{X}^H := \text{cl}(\tau_d(X))$ the *horofunction compactification* of (X, d) . The *horofunction boundary* of (X, d) is defined by

$$(2.2) \quad \partial_H X := \overline{X}^H \setminus \tau_d(X).$$

The elements of $\partial_H X$ are called *horofunctions* for the metric d on X . For

each $r \in \mathbb{R}$, the sublevel set $\mathcal{H}(h, r) := \{x \in X \mid h(x) \leq r\}$ is called a *horoball* centered at $h \in \partial_H X$.

REMARK 2.2. The mapping $y \mapsto \tau_d(y)$ is injective and continuous in the product topology. If (X, d) is *proper*, i.e., every closed ball is compact, then the mapping $y \mapsto \tau_d(y)$ defines an embedding $X \hookrightarrow \overline{X}^H$. By identifying X with $\tau_d(X)$, the horofunction boundary (2.2) becomes $\partial_H X = \overline{X}^H \setminus X$. The choice of the base point $b \in X$ is irrelevant, in the sense that horofunction boundaries of (X, d) for different base points are homeomorphic. We refer to [1, 3, 16] for further details.

REMARK 2.3. If X is a normed space with norm $\|\cdot\|$, then we choose the base point $b = 0 \in X$, and so (2.1) becomes $\tau(y)(x) = \|x - y\| - \|y\|$. Moreover, if X is finite-dimensional, then $(X, \|\cdot\|)$ is a proper metric space and hence any $h \in \overline{X}^H$ can be written as $h(x) = \lim_{n \rightarrow \infty} \tau(y^n)(x)$ for all $x \in X$ and for some sequence $\{y^n\}_{n \in \mathbb{N}}$ in X .

It is well-known that the horofunction boundary of $\ell_1(\{1\}, \mathbb{R}) := (\mathbb{R}, |\cdot|)$ has exactly two elements. More precisely, by considering unbounded sequences $\{y^n\}_{n \in \mathbb{N}}$ of real numbers, one obtains

$$(2.3) \quad \partial_H \ell_1(\{1\}, \mathbb{R}) = \{x \mapsto h_\epsilon(x) = \epsilon x \mid \epsilon \in \{-1, +1\}\}.$$

In the following sections we describe the horofunction boundary of $\ell_p(\mathcal{N}, \mathbb{R})$ for all $1 \leq p \leq \infty$. Throughout, we shall denote $\tau_p(y)(x) = \|x - y\|_p - \|y\|_p$, where $x, y \in \ell_p(\mathcal{N}, \mathbb{R})$.

3. The horofunction boundary of $\ell_1(\mathcal{N}, \mathbb{R})$

LEMMA 3.1. *Let $N \geq 2$ and $\mathcal{N} = \{1, \dots, N\}$. Let $\{y^n\}_{n \in \mathbb{N}}$ be a sequence in $\ell_1(\mathcal{N}, \mathbb{R})$ such that $\|y^n\|_1 \rightarrow \infty$ as $n \rightarrow \infty$. Then there exists $\emptyset \subsetneq \mathcal{I} \subseteq \mathcal{N}$ such that the sequence of functions $\{\tau_1(y^n)\}_{n \in \mathbb{N}}$ has a subsequence which converges pointwise to the function*

$$x \mapsto h_{\epsilon, \mu}^{\mathcal{I}}(x) := \sum_{i \in \mathcal{I}} \epsilon_i x_i + \sum_{i \in \mathcal{N} \setminus \mathcal{I}} (|x_i - \mu_i| - |\mu_i|),$$

where $\epsilon = (\epsilon_i)_i \in \{-1, +1\}^{\mathcal{I}}$ and $\mu = (\mu_i)_i \in \mathbb{R}^{\mathcal{N} \setminus \mathcal{I}}$.

Proof. Let $\{y^n\}_{n \in \mathbb{N}}$ be a sequence in $\ell_1(\mathcal{N}, \mathbb{R})$ such that $\|y^n\|_1 \rightarrow \infty$, as $n \rightarrow \infty$. By taking subsequences, we can find $\emptyset \subsetneq \mathcal{I} \subseteq \mathcal{N}$ such that $|y_i^n| \rightarrow \infty$, as $n \rightarrow \infty$, for all $i \in \mathcal{I}$, and $\{y_i^n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ is bounded for all $i \in \mathcal{N} \setminus \mathcal{I}$. By applying Cantor's diagonal argument and (2.3), we find

a further subsequence such that for every $x \in \ell_1(\mathcal{N}, \mathbb{R})$,

$$\begin{aligned} \tau_1(y^n)(x) &= \sum_{i \in \mathcal{N}} |x_i - y_i^n| - \sum_{i \in \mathcal{N}} |y_i^n| \\ &= \sum_{i \in \mathcal{I}} (|x_i - y_i^n| - |y_i^n|) + \sum_{i \in \mathcal{N} \setminus \mathcal{I}} (|x_i - y_i^n| - |y_i^n|) \\ &\xrightarrow{n \rightarrow \infty} \sum_{i \in \mathcal{I}} \epsilon_i x_i + \sum_{i \in \mathcal{N} \setminus \mathcal{I}} (|x_i - \mu_i| - |\mu_i|), \end{aligned}$$

where $\epsilon = (\epsilon_i)_i \in \{-1, +1\}^{\mathcal{I}}$ and $\mu = (\mu_i)_i \in \mathbb{R}^{\mathcal{N} \setminus \mathcal{I}}$. ■

THEOREM 3.2. *Let $N \geq 2$ and $\mathcal{N} = \{1, \dots, N\}$. The horofunction boundary of the metric space $\ell_1(\mathcal{N}, \mathbb{R})$ is given by*

$$(3.1) \quad \partial_H \ell_1(\mathcal{N}, \mathbb{R}) = \left\{ x \mapsto h_{\epsilon, \mu}^{\mathcal{I}}(x) \mid \begin{array}{l} \emptyset \subsetneq \mathcal{I} \subseteq \mathcal{N}, \epsilon \in \{-1, +1\}^{\mathcal{I}}, \\ \mu \in \mathbb{R}^{\mathcal{N} \setminus \mathcal{I}} \end{array} \right\},$$

where $h_{\epsilon, \mu}^{\mathcal{I}}(x) = \sum_{i \in \mathcal{I}} \epsilon_i x_i + \sum_{i \in \mathcal{N} \setminus \mathcal{I}} (|x_i - \mu_i| - |\mu_i|)$ for all $x \in \ell_1(\mathcal{N}, \mathbb{R})$.

Proof. Suppose that $h \in \partial_H \ell_1(\mathcal{N}, \mathbb{R})$. Then there exists a sequence $\{y^n\}_{n \in \mathbb{N}}$ in $\ell_1(\mathcal{N}, \mathbb{R})$ with $\|y^n\|_1 \rightarrow \infty$ such that $\{\tau_1(y^n)\}_{n \in \mathbb{N}}$ converges pointwise to h as $n \rightarrow \infty$. By Lemma 3.1 there exist $\emptyset \subsetneq \mathcal{I} \subseteq \mathcal{N}$, $\epsilon \in \{-1, +1\}^{\mathcal{I}}$ and $\mu \in \mathbb{R}^{\mathcal{N} \setminus \mathcal{I}}$ such that there is a subsequence $\{\tau_1(y^{n_k})\}_k$ that converges pointwise to $h_{\epsilon, \mu}^{\mathcal{I}}$ as $k \rightarrow \infty$. Therefore $h = h_{\epsilon, \mu}^{\mathcal{I}}$ and so $\partial_H \ell_1(\mathcal{N}, \mathbb{R})$ is contained in the set on the right-hand side of (3.1).

For the other inclusion, assume that \mathcal{I} is any nonempty subset of \mathcal{N} . Let $\epsilon \in \{-1, +1\}^{\mathcal{I}}$ and let $\mu \in \mathbb{R}^{\mathcal{N} \setminus \mathcal{I}}$. We will show that the function $h_{\epsilon, \mu}^{\mathcal{I}}$ belongs to $\overline{\ell_1(\mathcal{N}, \mathbb{R})}^H \setminus \tau_1(\ell_1(\mathcal{N}, \mathbb{R}))$. Indeed, for each n define $y^n = (y_i^n)_{i \in \mathcal{N}}$ in $\ell_1(\mathcal{N}, \mathbb{R})$ by

$$(3.2) \quad y_i^n = \begin{cases} -\epsilon_i n, & i \in \mathcal{I}, \\ \mu_i, & i \in \mathcal{N} \setminus \mathcal{I}. \end{cases}$$

Then for every $x \in \ell_1(\mathcal{N}, \mathbb{R})$ we have

$$\begin{aligned} \tau_1(y^n)(x) &= \sum_{i \in \mathcal{I}} (|x_i - y_i^n| - |y_i^n|) + \sum_{i \in \mathcal{N} \setminus \mathcal{I}} (|x_i - y_i^n| - |y_i^n|) \\ &= \sum_{i \in \mathcal{I}} (|x_i + \epsilon_i n| - n) + \sum_{i \in \mathcal{N} \setminus \mathcal{I}} (|x_i - \mu_i| - |\mu_i|) \\ &\xrightarrow{n \rightarrow \infty} \sum_{i \in \mathcal{I}} \epsilon_i x_i + \sum_{i \in \mathcal{N} \setminus \mathcal{I}} (|x_i - \mu_i| - |\mu_i|) = h_{\epsilon, \mu}^{\mathcal{I}}(x). \end{aligned}$$

Therefore $h_{\epsilon, \mu}^{\mathcal{I}} \in \overline{\ell_1(\mathcal{N}, \mathbb{R})}^H$. It remains to show that $h_{\epsilon, \mu}^{\mathcal{I}}$ is not an element of $\tau_1(\ell_1(\mathcal{N}, \mathbb{R}))$. Suppose the contrary, so there exists $z \in \ell_1(\mathcal{N}, \mathbb{R})$ such that

$h_{\epsilon, \mu}^{\mathcal{I}} = \tau_1(z)$. It follows by (3.2) that

$$h_{\epsilon, \mu}^{\mathcal{I}}(y^n) = -n|\mathcal{I}| - \sum_{i \in \mathcal{N} \setminus \mathcal{I}} |\mu_i| \xrightarrow{n \rightarrow \infty} -\infty.$$

However, by (2.1) we know that $\tau_1(z)$ is bounded from below by $-\|z\|_1$, and hence

$$\liminf_{n \rightarrow \infty} \tau_1(z)(y^n) \geq -\|z\|_1 > -\infty,$$

which is a contradiction. Therefore $h_{\epsilon, \mu}^{\mathcal{I}}$ belongs to $\partial_H \ell_1(\mathcal{N}, \mathbb{R})$, that is, every element of the set on the right-hand side of (3.1) is a horofunction on $\ell_1(\mathcal{N}, \mathbb{R})$. ■

4. The horofunction boundary of $\ell_p(\mathcal{N}, \mathbb{R})$ for $1 < p < \infty$. Recall that a normed space $(X, \|\cdot\|)$ is called *uniformly convex* if for every $\epsilon \in]0, 2]$ there exists $\delta > 0$ such that $\|x + y\| \leq 2(1 - \delta)$ whenever $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \epsilon$. A well-known result due to Clarkson [6] is that L_p and ℓ_p spaces are uniformly convex for $1 < p < \infty$. It will be convenient to use the following equivalent characterization of uniform convexity.

PROPOSITION 4.1 ([8, p. 287]). *A Banach space $(X, \|\cdot\|)$ is uniformly convex if and only if $\|x_n - y_n\| \rightarrow 0$, as $n \rightarrow \infty$, whenever $x_n, y_n \in X$ with $\|x_n\| \leq 1$, $\|y_n\| \leq 1$ for all $n \in \mathbb{N}$, and $\|x_n + y_n\| \rightarrow 2$ as $n \rightarrow \infty$.*

LEMMA 4.2. *Let $p, q \in]1, \infty[$ be such that $p^{-1} + q^{-1} = 1$. Let $\{y^n\}_{n \in \mathbb{N}}$ be a sequence in $\ell_p(\mathcal{N}, \mathbb{R})$ such that $\|y^n\|_p \rightarrow \infty$ as $n \rightarrow \infty$. Then there exists $\mu \in \ell_q(\mathcal{N}, \mathbb{R})$ with $\|\mu\|_q = 1$ for which the sequence of functions $\{\tau_p(y^n)\}_{n \in \mathbb{N}}$ has a subsequence converging pointwise to the function*

$$x \mapsto h_\mu(x) := - \sum_{i \in \mathcal{N}} \mu_i x_i.$$

Proof. We may assume without loss of generality that $y^n \neq 0$, and define $w^n := y^n / \|y^n\|_p$ for all n . By compactness of the unit sphere of $\ell_p(\mathcal{N}, \mathbb{R})$, it follows that there exists a subsequence $\{w^{n_k}\}_k$ that converges, as $k \rightarrow \infty$, to some $w \in \ell_p(\mathcal{N}, \mathbb{R})$ with $\|w\|_p = 1$. Therefore, by ℓ_p/ℓ_q -duality there exists a unique $\mu \in \ell_q(\mathcal{N}, \mathbb{R})$ with $\|\mu\|_q = 1$ such that $\langle \mu, w \rangle = 1$. Now, let $x \in \ell_p(\mathcal{N}, \mathbb{R})$ and for each k define

$$(4.1) \quad z^k := \frac{y^{n_k} - x}{\|x - y^{n_k}\|_p} = \frac{-x}{\|x - y^{n_k}\|_p} + \frac{\|y^{n_k}\|_p}{\|x - y^{n_k}\|_p} w^{n_k}.$$

For each k we have $\|z^k\|_p = 1$, and hence by ℓ_p/ℓ_q -duality there exists $\varphi^k \in \ell_q(\mathcal{N}, \mathbb{R})$ with $\|\varphi^k\|_q = 1$ such that $\langle \varphi^k, z^k \rangle = 1$. By applying the assumption $\|y^{n_k}\|_p \rightarrow \infty$ to (4.1), we obtain $\|z^k - w\|_p \rightarrow 0$ as $k \rightarrow \infty$.

Consequently,

$$2 = \|\varphi^k\|_q + \|\mu\|_q \geq \|\varphi^k + \mu\|_q \geq \langle \varphi^k + \mu, z^k \rangle = 1 + \langle \mu, z^k \rangle \xrightarrow[k \rightarrow \infty]{} 2,$$

and hence, by Proposition 4.1, we have $\|\varphi^k - \mu\|_q \rightarrow 0$ as $k \rightarrow \infty$. On the other hand, by evaluating each dual pairing of φ^k at z^k in (4.1) we obtain

$$\|x - y^{n_k}\|_p = \langle \varphi^k, -x \rangle + \|y^{n_k}\|_p \langle \varphi^k, w^{n_k} \rangle.$$

Therefore,

$$\begin{aligned} \tau_p(y^{n_k})(x) &= \|x - y^{n_k}\|_p - \|y^{n_k}\|_p = \langle \varphi^k, -x \rangle + \|y^{n_k}\|_p \langle \varphi^k, w^{n_k} \rangle - \|y^{n_k}\|_p \\ &\xrightarrow[k \rightarrow \infty]{} -\langle \mu, x \rangle = h_\mu(x). \quad \blacksquare \end{aligned}$$

THEOREM 4.3. *Let $p, q \in]1, \infty[$ be such that $p^{-1} + q^{-1} = 1$. The horofunction boundary of the metric space $\ell_p(\mathcal{N}, \mathbb{R})$ is given by*

$$(4.2) \quad \partial_H \ell_p(\mathcal{N}, \mathbb{R}) = \{x \mapsto h_\mu(x) \mid \mu \in \ell_q(\mathcal{N}, \mathbb{R}), \|\mu\|_q = 1\},$$

where $h_\mu(x) = -\sum_{i \in \mathcal{N}} \mu_i x_i$ for all $x \in \ell_p(\mathcal{N}, \mathbb{R})$.

Proof. If $h \in \partial_H \ell_p(\mathcal{N}, \mathbb{R})$, then there exists a sequence $\{y^n\}_{n \in \mathbb{N}}$ with $\|y^n\|_p \rightarrow \infty$ such that h is the pointwise limit of the sequence $\{\tau_p(y^n)\}_{n \in \mathbb{N}}$. By Lemma 4.2, there exists $\mu \in \ell_q(\mathcal{N}, \mathbb{R})$ with $\|\mu\|_q = 1$ such that along subsequences $\tau_p(y^n)(x)$ converges to $h_\mu(x) = -\sum_{i \in \mathcal{N}} \mu_i x_i$ for all $x \in \ell_p(\mathcal{N}, \mathbb{R})$. Therefore $h = h_\mu$ and so $\partial_H \ell_p(\mathcal{N}, \mathbb{R})$ is contained in the set on the right-hand side of (4.2).

On the other hand, if $\mu \in \ell_q(\mathcal{N}, \mathbb{R})$ with $\|\mu\|_q = 1$, then by ℓ_p/ℓ_q -duality there exists $w \in \ell_p(\mathcal{N}, \mathbb{R})$ with $\|w\|_p = 1$ such that $\sum_{i \in \mathcal{N}} \mu_i w_i = 1$. Let $y^n = nw$ for all n . Then, by proceeding as in Lemma 4.2, it follows that $\tau_p(y^n)(x) = \|x - nw\|_p - n$ converges to $h_\mu(x) = -\sum_{i \in \mathcal{N}} \mu_i x_i$ for all $x \in \ell_p(\mathcal{N}, \mathbb{R})$. That is, h_μ belongs to $\overline{\ell_p(\mathcal{N}, \mathbb{R})}^H$. However, note that $h_\mu(y^n) = -n$ for all n . Therefore, since for any $z \in \ell_p(\mathcal{N}, \mathbb{R})$ the function $\tau_p(z)$ is bounded from below, we must have $h_\mu \in \partial_H \ell_p(\mathcal{N}, \mathbb{R})$. That is, every element of the set on the right-hand side of (4.2) is a horofunction on $\ell_p(\mathcal{N}, \mathbb{R})$. \blacksquare

REMARK 4.4. Lemma 4.2 and Theorem 4.3 hold for every finite-dimensional uniformly convex Banach space.

5. The horofunction boundary of $\ell_\infty(\mathcal{N}, \mathbb{R})$. It will be convenient and helpful to consider the *top function* \mathbf{t} and the *bottom function* \mathbf{b} defined on $\mathbb{R}^{\mathcal{N}}$ by

$$\mathbf{t}(x) := \max_{i \in \mathcal{N}} x_i, \quad \mathbf{b}(x) := \min_{i \in \mathcal{N}} x_i.$$

These functions simplify notations significantly when proving Lemma 5.1 and Theorem 5.2 in this section as well as Lemma 6.1, Theorem 6.2, and

Corollary 6.3 in Section 6. The norm $\|\cdot\|_\infty$ on $\mathbb{R}^\mathcal{N}$ can be redefined as

$$(5.1) \quad \|x\|_\infty = \max\{\mathfrak{t}(x), -\mathfrak{b}(x)\}.$$

The standard cone $\mathbb{R}_+^\mathcal{N}$ of $\mathbb{R}^\mathcal{N}$ is defined by

$$\mathbb{R}_+^\mathcal{N} := \{x \in \mathbb{R}^\mathcal{N} \mid x_i \geq 0, \forall i \in \mathcal{N}\}.$$

We denote by $\mathbb{R}_{>0}^\mathcal{N}$ the interior of $\mathbb{R}_+^\mathcal{N}$. The boundary $\partial\mathbb{R}_+^\mathcal{N}$ of $\mathbb{R}_+^\mathcal{N}$ is the set $\mathbb{R}_+^\mathcal{N} \setminus \mathbb{R}_{>0}^\mathcal{N}$. We shall denote by $\mathbb{1}$ the element of $\mathbb{R}^\mathcal{N}$ given by $\mathbb{1} = (1, \dots, 1)$. It follows that $x - \mathfrak{b}(x)\mathbb{1}$ and $\mathfrak{t}(x)\mathbb{1} - x$ are both elements of $\partial\mathbb{R}_+^\mathcal{N}$ for all $x \in \mathbb{R}^\mathcal{N}$. The mapping $\text{Exp} : \mathbb{R}^\mathcal{N} \rightarrow \mathbb{R}_{>0}^\mathcal{N}$ is defined by $\text{Exp}(x)_i := e^{x_i}$ for all $i \in \mathcal{N}$. Similarly, the mapping $\text{Log} : \mathbb{R}_{>0}^\mathcal{N} \rightarrow \mathbb{R}^\mathcal{N}$ is defined by $\text{Log}(u)_i := \log u_i$ for all $i \in \mathcal{N}$.

The *Hadamard product* of any two elements $x = (x_i)_{i \in \mathcal{N}}$ and $y = (y_i)_{i \in \mathcal{N}}$ of $\mathbb{R}^\mathcal{N}$, denoted by $x \odot y$, is another element of $\mathbb{R}^\mathcal{N}$ defined by $(x \odot y)_i := x_i y_i$ for all $i \in \mathcal{N}$. For every $x = (x_i)_{i \in \mathcal{N}}$ in $\mathbb{R}_{>0}^\mathcal{N}$ we shall denote by x^{-1} the element of $\mathbb{R}_{>0}^\mathcal{N}$ defined by $(x^{-1})_i := 1/x_i$ for all $i \in \mathcal{N}$.

Using the notations introduced above, it readily follows that

$$(5.2) \quad \mathfrak{t}(\text{Exp}(x) \odot \text{Exp}(y)) = \exp \mathfrak{t}(x + y) \quad \text{for all } x, y \in \mathbb{R}^\mathcal{N},$$

$$(5.3) \quad \mathfrak{t}(\text{Log}(x) - \text{Log}(y)) = \log \mathfrak{t}(x \odot y^{-1}) \quad \text{for all } x, y \in \mathbb{R}_{>0}^\mathcal{N}.$$

Note that $\|x\|_\infty = \max\{\mathfrak{t}(x), \mathfrak{t}(-x)\} = \mathfrak{t}(x, -x)$ for all $x \in \mathbb{R}^\mathcal{N}$. Therefore, the mapping $y \mapsto \tau_\infty(y)$ becomes

$$\begin{aligned} \tau_\infty(y)(x) &= \|x - y\|_\infty - \|y\|_\infty = \mathfrak{t}(x - y, -x + y) - \|y\|_\infty \\ &= \mathfrak{t}(x - y - \|y\|_\infty \mathbb{1}, -x + y - \|y\|_\infty \mathbb{1}). \end{aligned}$$

For any $x = (x_i)_{i \in \mathcal{N}} \in \mathbb{R}^\mathcal{N}$ and any nonempty subset \mathcal{I} of \mathcal{N} we shall write $x_\mathcal{I} = (x_i)_{i \in \mathcal{I}}$.

LEMMA 5.1. *Let $\{y^n\}_{n \in \mathbb{N}}$ be a sequence in $\ell_\infty(\mathcal{N}, \mathbb{R})$ such that $\|y^n\|_\infty \rightarrow \infty$ as $n \rightarrow \infty$. Then $\{\tau_\infty(y^n)\}_{n \in \mathbb{N}}$ has a subsequence which converges pointwise to the function*

$$x \mapsto h_{\mu, \nu}^{\mathcal{I}, \mathcal{J}}(x) := \mathfrak{t}(x_\mathcal{I} - \mu, -x_\mathcal{J} - \nu),$$

where $\emptyset \subseteq \mathcal{I}, \mathcal{J} \subseteq \mathcal{N}$ with $\mathcal{I} \cap \mathcal{J} = \emptyset$, $\mathcal{I} \cup \mathcal{J} \neq \emptyset$, and $\mu \in \mathbb{R}_+^\mathcal{I}$, $\nu \in \mathbb{R}_+^\mathcal{J}$ with $\mathfrak{b}(\mu, \nu) = 0$.

Proof. For each n , define

$$u^n = \text{Exp}(-y^n - \|y^n\|_\infty \mathbb{1}), \quad v^n = \text{Exp}(y^n - \|y^n\|_\infty \mathbb{1}).$$

It follows that $(u^n, v^n) \in \mathbb{R}_+^\mathcal{N} \times \mathbb{R}_+^\mathcal{N}$ with $\mathfrak{t}(u^n, v^n) = 1$ for all n . Therefore, there exists a subsequence $\{(u^{n_k}, v^{n_k})\}_k$ which converges, as $k \rightarrow \infty$, to $(u, v) \in \mathbb{R}_+^\mathcal{N} \times \mathbb{R}_+^\mathcal{N}$ with $\mathfrak{t}(u, v) = 1$. Furthermore, note that $u^{n_k} \odot v^{n_k} = \text{Exp}(-2\|y^{n_k}\|_\infty \mathbb{1})$ for all k . Hence by taking the limit as $k \rightarrow \infty$, we obtain $u \odot v = 0\mathbb{1}$. Consequently, there exist $\emptyset \subseteq \mathcal{I}, \mathcal{J} \subseteq \mathcal{N}$ with $\mathcal{I} \cap \mathcal{J} = \emptyset$ and

$\mathcal{I} \cup \mathcal{J} \neq \emptyset$ such that $0 < u_i \leq 1$ for all $i \in \mathcal{I}$, $0 < v_j \leq 1$, for all $j \in \mathcal{J}$ with $\mathbf{t}(u_{\mathcal{I}}, v_{\mathcal{J}}) = 1$. Now, by letting $\mu = -\text{Log}(u_{\mathcal{I}})$ and $\nu = -\text{Log}(v_{\mathcal{J}})$, it follows that $\mu \in \mathbb{R}_+^{\mathcal{I}}$, $\nu \in \mathbb{R}_+^{\mathcal{J}}$ with $\mathbf{b}(\mu, \nu) = 0$. Finally, let $x \in \ell_{\infty}(\mathcal{N}, \mathbb{R})$; then by (5.2) we have

$$\begin{aligned} \lim_{k \rightarrow \infty} \tau_{\infty}(y^{n_k})(x) &= \lim_{k \rightarrow \infty} \mathbf{t}(x - y^{n_k} - \|y^{n_k}\|_{\infty} \mathbb{1}, -x + y^{n_k} - \|y^{n_k}\|_{\infty} \mathbb{1}) \\ &= \lim_{k \rightarrow \infty} \log \mathbf{t}(\text{Exp}(x) \odot u^{n_k}, \text{Exp}(-x) \odot v^{n_k}) \\ &= \log \mathbf{t}(\text{Exp}(x) \odot u, \text{Exp}(-x) \odot v) \\ &= \log \mathbf{t}(\text{Exp}(x_{\mathcal{I}}) \odot u_{\mathcal{I}}, \text{Exp}(-x_{\mathcal{J}}) \odot v_{\mathcal{J}}) \\ &= \mathbf{t}(x_{\mathcal{I}} - \mu, -x_{\mathcal{J}} - \nu). \blacksquare \end{aligned}$$

Let $\overline{\mathbb{R}}$ denote the extended set of real numbers $\mathbb{R} \cup \{-\infty, \infty\}$. The top function \mathbf{t} and bottom function \mathbf{b} can be redefined on $\overline{\mathbb{R}}^{\mathcal{N}}$ according to the natural order in $\overline{\mathbb{R}}$. Let $\overline{\mathbb{R}}_+^{\mathcal{N}}$ denote the set

$$\overline{\mathbb{R}}_+^{\mathcal{N}} = \{x \in \overline{\mathbb{R}}^{\mathcal{N}} \mid 0 \leq x_i \leq \infty, \forall i = 1, \dots, N\} = [0, \infty]^{\mathcal{N}}.$$

THEOREM 5.2. *The horofunction boundary of the metric space $\ell_{\infty}(\mathcal{N}, \mathbb{R})$ is given by*

$$(5.4) \quad \partial_H \ell_{\infty}(\mathcal{N}, \mathbb{R}) = \left\{ x \mapsto h_{\overline{\mu}, \overline{\nu}}(x) \mid \begin{array}{l} \overline{\mu}, \overline{\nu} \in \overline{\mathbb{R}}_+^{\mathcal{N}}, \mathbf{b}(\overline{\mu}, \overline{\nu}) = 0, \\ \overline{\mu} + \overline{\nu} = \infty \mathbb{1} \end{array} \right\},$$

where $h_{\overline{\mu}, \overline{\nu}}(x) = \mathbf{t}(x - \overline{\mu}, -x - \overline{\nu})$ for all $x \in \ell_{\infty}(\mathcal{N}, \mathbb{R})$.

Proof. Suppose that $h \in \partial_H \ell_{\infty}(\mathcal{N}, \mathbb{R})$. Then there exists a sequence $\{y^n\}_{n \in \mathbb{N}}$ in $\ell_{\infty}(\mathcal{N}, \mathbb{R})$ with $\|y^n\|_{\infty} \rightarrow \infty$ such that $\tau_{\infty}(y^n)$ converges pointwise to h as $n \rightarrow \infty$. Let $x \in \ell_{\infty}(\mathcal{N}, \mathbb{R})$. By Lemma 5.1 there exist $\emptyset \subseteq \mathcal{I}$, $\mathcal{J} \subseteq \mathcal{N}$ with $\mathcal{I} \cap \mathcal{J} = \emptyset$, $\mathcal{I} \cup \mathcal{J} \neq \emptyset$, and $\mu \in \mathbb{R}_+^{\mathcal{I}}$, $\nu \in \mathbb{R}_+^{\mathcal{J}}$ with $\mathbf{b}(\mu, \nu) = 0$, such that for some subsequence $\{y^{n_k}\}_k$ we have

$$h(x) = \lim_{k \rightarrow \infty} \tau_{\infty}(y^{n_k})(x) = \mathbf{t}(x_{\mathcal{I}} - \mu, -x_{\mathcal{J}} - \nu).$$

Finally, by letting

$$\overline{\mu}_i = \begin{cases} \mu_i, & i \in \mathcal{I}, \\ \infty, & i \in \mathcal{N} \setminus \mathcal{I}, \end{cases} \quad \overline{\nu}_i = \begin{cases} \nu_i, & i \in \mathcal{J}, \\ \infty, & i \in \mathcal{N} \setminus \mathcal{J} \end{cases}$$

we get $\overline{\mu}, \overline{\nu} \in \overline{\mathbb{R}}_+^{\mathcal{N}}$ with $\overline{\mu} + \overline{\nu} = \infty \mathbb{1}$, and $\mathbf{b}(\overline{\mu}, \overline{\nu}) = 0$. Hence, $h(x) = h_{\overline{\mu}, \overline{\nu}}(x)$ and so $\partial_H \ell_{\infty}(\mathcal{N}, \mathbb{R})$ is contained in the set on the right-hand side of (5.4).

Now, we need to show that given $\overline{\mu}, \overline{\nu} \in \overline{\mathbb{R}}_+^{\mathcal{N}}$ with $\overline{\mu} + \overline{\nu} = \infty \mathbb{1}$ and $\mathbf{b}(\overline{\mu}, \overline{\nu}) = 0$, the function $x \mapsto h_{\overline{\mu}, \overline{\nu}}(x)$ is a horofunction on $\ell_{\infty}(\mathcal{N}, \mathbb{R})$. First we show that it belongs to $\overline{\ell_{\infty}(\mathcal{N}, \mathbb{R})}^H$. Indeed, let $(y^n)_n$ be the sequence in

$\ell_\infty(\mathcal{N}, \mathbb{R})$ given by

$$y_i^n = \begin{cases} -n + \bar{\mu}_i, & \bar{\mu}_i < \infty, \\ n - \bar{\nu}_i, & \bar{\nu}_i < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Let $x \in \ell_\infty(\mathcal{N}, \mathbb{R})$. Then

$$\tau_\infty(y^n)(x) = \|x - y^n\|_\infty - \|y^n\|_\infty \xrightarrow{n \rightarrow \infty} \mathbf{t}(x - \bar{\mu}, -x - \bar{\nu}) = h_{\bar{\mu}, \bar{\nu}}(x).$$

It remains to show that $h_{\bar{\mu}, \bar{\nu}}$ is not an element of $\tau_\infty(\ell_\infty(\mathcal{N}, \mathbb{R}))$. Suppose the contrary, so there exists $z \in \ell_\infty(\mathcal{N}, \mathbb{R})$ such that

$$h_{\bar{\mu}, \bar{\nu}}(x) = \mathbf{t}(x - \bar{\mu}, -x - \bar{\nu}) = \tau_\infty(z)(x).$$

For each k , define

$$x_i^k = \begin{cases} \bar{\mu}_i, & \bar{\mu}_i < \infty, \\ -\bar{\nu}_i, & \bar{\nu}_i < \infty, \\ -k & \text{otherwise.} \end{cases}$$

Then $h_{\bar{\mu}, \bar{\nu}}(x^k) = \mathbf{t}(x^k - \bar{\mu}, -x^k - \bar{\nu}) = 0$ for all k . However,

$$\tau_\infty(z)(x^k) = \|x^k - z\|_\infty - \|z\|_\infty \xrightarrow{k \rightarrow \infty} 0,$$

which is a contradiction. Therefore $h_{\bar{\mu}, \bar{\nu}} \in \partial_H \ell_\infty(\mathcal{N}, \mathbb{R})$ and so the other inclusion holds. ■

6. The horofunction boundary of $\ell_{\text{var}}(\mathcal{N}, \mathbb{R})$. We define the *variation norm* on $\mathbb{R}^{\mathcal{N}}$ by

$$(6.1) \quad \|x\|_{\text{var}} := \mathbf{t}(x) - \mathbf{b}(x),$$

where \mathbf{t} and \mathbf{b} are, respectively, the top and bottom functions introduced in Section 5. In fact, $\|\cdot\|_{\text{var}}$ is a pseudo-norm on $\mathbb{R}^{\mathcal{N}}$, as $\|x\|_{\text{var}} = 0$ if and only if $x = \lambda \mathbb{1}$ for some $\lambda \in \mathbb{R}$. Moreover $\|x + \lambda \mathbb{1}\|_{\text{var}} = \|x\|_{\text{var}}$ for all $x \in \mathbb{R}^{\mathcal{N}}$. Hence $\|\cdot\|_{\text{var}}$ is a norm on the quotient vector space $\mathbb{R}^{\mathcal{N}}/\mathbb{R}\mathbb{1}$. By (6.1), the mapping $y \mapsto \tau_{\text{var}}(y)$ becomes

$$(6.2) \quad \begin{aligned} \tau_{\text{var}}(y)(x) &= \|x - y\|_{\text{var}} - \|y\|_{\text{var}} = \mathbf{t}(x - y) - \mathbf{b}(x - y) - \mathbf{t}(y) + \mathbf{b}(y) \\ &= \mathbf{t}(x - y + \mathbf{b}(y)\mathbb{1}) - \mathbf{b}(x - y + \mathbf{t}(y)\mathbb{1}). \end{aligned}$$

LEMMA 6.1. *Let $N \geq 2$ and $\mathcal{N} = \{1, \dots, N\}$. Let $\{y^n\}_{n \in \mathbb{N}}$ be a sequence in $\ell_{\text{var}}(\mathcal{N}, \mathbb{R})$ such that $\|y^n\|_{\text{var}} \rightarrow \infty$ as $n \rightarrow \infty$. Then $\{\tau_{\text{var}}(y^n)\}_{n \in \mathbb{N}}$ has a subsequence which converges pointwise to the function*

$$x \mapsto h_{\mu, \nu}^{\mathcal{I}, \mathcal{J}}(x) := \mathbf{t}(x_{\mathcal{I}} - \mu) - \mathbf{b}(x_{\mathcal{J}} + \nu),$$

where $\emptyset \subsetneq \mathcal{I}, \mathcal{J} \subsetneq \mathcal{N}$ with $\mathcal{I} \cap \mathcal{J} = \emptyset$, and $\mu \in \partial \mathbb{R}_+^{\mathcal{I}}, \nu \in \partial \mathbb{R}_+^{\mathcal{J}}$.

Proof. For each n , define

$$u^n = \text{Exp}(\mathbf{b}(y^n)\mathbb{1} - y^n), \quad v^n = \text{Exp}(y^n - \mathbf{t}(y^n)\mathbb{1}).$$

It follows that $(u^n, v^n) \in \mathbb{R}_+^{\mathcal{N}} \times \mathbb{R}_+^{\mathcal{N}}$ with $\mathbf{t}(u^n) = 1$, $\mathbf{t}(v^n) = 1$ for all n . Hence, there exists a subsequence $\{(u^{n_k}, v^{n_k})\}_k$ which converges, as $k \rightarrow \infty$, to some $(u, v) \in \mathbb{R}_+^{\mathcal{N}} \times \mathbb{R}_+^{\mathcal{N}}$ with $\mathbf{t}(u) = 1$, $\mathbf{t}(v) = 1$. Let $x \in \ell_{\text{var}}(\mathcal{N}, \mathbb{R})$. By (5.2), it follows that for every k ,

- (i) $\log \mathbf{t}(\text{Exp}(x) \odot u^{n_k}) = \mathbf{t}(x - y^{n_k} + \mathbf{b}(y^{n_k})\mathbb{1})$,
- (ii) $\log \mathbf{t}(\text{Exp}(-x) \odot v^{n_k}) = \mathbf{t}(-x + y^{n_k} - \mathbf{t}(y^{n_k})\mathbb{1}) = -\mathbf{b}(x - y^{n_k} + \mathbf{t}(y^{n_k})\mathbb{1})$.

Therefore, by (6.2),

$$\begin{aligned} \tau_{\text{var}}(y^{n_k})(x) &= \mathbf{t}(x - y^{n_k} + \mathbf{b}(y^{n_k})\mathbb{1}) - \mathbf{b}(x - y^{n_k} + \mathbf{t}(y^{n_k})\mathbb{1}), \\ &= \log \mathbf{t}(\text{Exp}(x) \odot u^{n_k}) + \log \mathbf{t}(\text{Exp}(-x) \odot v^{n_k}), \\ &\xrightarrow[k \rightarrow \infty]{} \log \mathbf{t}(\text{Exp}(x) \odot u) + \log \mathbf{t}(\text{Exp}(-x) \odot v). \end{aligned}$$

Also note that $u^{n_k} \odot v^{n_k} = \exp(-\|y^{n_k}\|_{\text{var}})\mathbb{1}$ for all k . Hence, by taking the limit as $k \rightarrow \infty$ we obtain $u \odot v = 0\mathbb{1}$. Consequently, there exist $\emptyset \subsetneq \mathcal{I}, \mathcal{J} \subsetneq \mathcal{N}$ with $\mathcal{I} \cap \mathcal{J} = \emptyset$ such that $0 < u_i \leq 1$ for all $i \in \mathcal{I}$, and $0 < v_j \leq 1$ for all $j \in \mathcal{J}$. Let $\mu = -\text{Log}(u_{\mathcal{I}})$ and let $\nu = -\text{Log}(v_{\mathcal{J}})$. Then $\mu \in \partial\mathbb{R}_+^{\mathcal{I}}$ and $\nu \in \partial\mathbb{R}_+^{\mathcal{J}}$. Therefore, by (5.2), it follows that

$$\begin{aligned} \lim_{k \rightarrow \infty} \tau_{\text{var}}(y^{n_k})(x) &= \log \mathbf{t}(\text{Exp}(x) \odot u) + \log \mathbf{t}(\text{Exp}(-x) \odot v) \\ &= \log \mathbf{t}(\text{Exp}(x_{\mathcal{I}}) \odot u_{\mathcal{I}}) + \log \mathbf{t}(\text{Exp}(-x_{\mathcal{J}}) \odot v_{\mathcal{J}}) \\ &= \mathbf{t}(x_{\mathcal{I}} - \mu) + \mathbf{t}(-x_{\mathcal{J}} - \nu) = \mathbf{t}(x_{\mathcal{I}} - \mu) - \mathbf{b}(x_{\mathcal{J}} + \nu) \\ &= h_{\mu, \nu}^{\mathcal{I}, \mathcal{J}}(x). \quad \blacksquare \end{aligned}$$

THEOREM 6.2. *Let $N \geq 2$ and $\mathcal{N} = \{1, \dots, N\}$. The horofunction boundary of the pseudo-normed space $\ell_{\text{var}}(\mathcal{N}, \mathbb{R})$ is given by*

$$(6.3) \quad \partial_H \ell_{\text{var}}(\mathcal{N}, \mathbb{R}) = \left\{ x \mapsto h_{\mu, \nu}^{\mathcal{I}, \mathcal{J}}(x) \mid \begin{array}{l} \emptyset \subsetneq \mathcal{I}, \mathcal{J} \subsetneq \mathcal{N}, \mathcal{I} \cap \mathcal{J} = \emptyset, \\ \mu \in \partial\mathbb{R}_+^{\mathcal{I}}, \nu \in \partial\mathbb{R}_+^{\mathcal{J}} \end{array} \right\},$$

where $h_{\mu, \nu}^{\mathcal{I}, \mathcal{J}}(x) = \mathbf{t}(x_{\mathcal{I}} - \mu) - \mathbf{b}(x_{\mathcal{J}} + \nu)$ for all $x \in \ell_{\text{var}}(\mathcal{N}, \mathbb{R})$.

Proof. Suppose that $h \in \partial_H \ell_{\text{var}}(\mathcal{N}, \mathbb{R})$. Then there exists a sequence $\{y^n\}_{n \in \mathbb{N}}$ in $\ell_{\text{var}}(\mathcal{N}, \mathbb{R})$ with $\|y^n\|_{\text{var}} \rightarrow \infty$ such that $\tau_{\text{var}}(y^n)$ converges pointwise to h as $n \rightarrow \infty$. By Lemma 6.1, there exist $\emptyset \subsetneq \mathcal{I}, \mathcal{J} \subsetneq \mathcal{N}$ with $\mathcal{I} \cap \mathcal{J} = \emptyset$, and $\mu \in \partial\mathbb{R}_+^{\mathcal{I}}$, $\nu \in \partial\mathbb{R}_+^{\mathcal{J}}$ such that there is a subsequence $\tau_{\text{var}}(y^{n_k})$ that converges pointwise to $h_{\mu, \nu}^{\mathcal{I}, \mathcal{J}}$ as $k \rightarrow \infty$. Therefore $h = h_{\mu, \nu}^{\mathcal{I}, \mathcal{J}}$ and so $\partial_H \ell_{\text{var}}(\mathcal{N}, \mathbb{R})$ is contained in the set on the right-hand side of (6.3).

Now, we need to show that for given $\emptyset \subsetneq \mathcal{I}, \mathcal{J} \subsetneq \mathcal{N}$ with $\mathcal{I} \cap \mathcal{J} = \emptyset$, and $\mu \in \partial\mathbb{R}_+^{\mathcal{I}}$, $\nu \in \partial\mathbb{R}_+^{\mathcal{J}}$, the function $h_{\mu, \nu}^{\mathcal{I}, \mathcal{J}}$ is a horofunction on $\ell_{\text{var}}(\mathcal{N}, \mathbb{R})$.

First we show that $h_{\mu,\nu}^{\mathcal{I},\mathcal{J}}$ belongs to $\overline{\ell_{\text{var}}(\mathcal{N}, \mathbb{R})}^H$. Let $(y^n)_n$ be the sequence in $\ell_{\text{var}}(\mathcal{N}, \mathbb{R})$ given by

$$y_i^n = \begin{cases} -n + \mu_i, & i \in \mathcal{I}, \\ n - \nu_i, & i \in \mathcal{J}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $x \in \ell_{\text{var}}(\mathcal{N}, \mathbb{R})$. Then by (6.2) we have

$$\tau_{\text{var}}(y^n)(x) = \|x - y^n\|_{\text{var}} - \|y^n\|_{\text{var}} \xrightarrow{n \rightarrow \infty} \mathfrak{t}(x_{\mathcal{I}} - \mu) - \mathfrak{b}(x_{\mathcal{J}} + \nu) = h_{\mu,\nu}^{\mathcal{I},\mathcal{J}}(x).$$

Hence $h_{\mu,\nu}^{\mathcal{I},\mathcal{J}}$ is an element of $\overline{\ell_{\text{var}}(\mathcal{N}, \mathbb{R})}^H$. It remains to show that $h_{\mu,\nu}^{\mathcal{I},\mathcal{J}}$ is not in $\tau_{\text{var}}(\ell_{\text{var}}(\mathcal{N}, \mathbb{R}))$. Suppose the contrary, so there exists $z \in \ell_{\text{var}}(\mathcal{N}, \mathbb{R})$ such that $h_{\mu,\nu}^{\mathcal{I},\mathcal{J}} = \tau_{\text{var}}(z)$. For each k , define

$$x_i^k = \begin{cases} \mu_i, & i \in \mathcal{I}, \\ -\nu_i, & i \in \mathcal{J}, \\ -k & \text{otherwise.} \end{cases}$$

Then $h_{\mu,\nu}^{\mathcal{I},\mathcal{J}}(x^k) = 0$ for all k . However,

$$\tau_{\text{var}}(z)(x^k) = \|x^k - z\|_{\text{var}} - \|z\|_{\text{var}} \xrightarrow{k \rightarrow \infty} 0,$$

which is a contradiction. Therefore $h_{\mu,\nu}^{\mathcal{I},\mathcal{J}}$ belongs to $\partial_H \ell_{\text{var}}(\mathcal{N}, \mathbb{R})$. ■

6.1. Hilbert's projective metric on $\mathbb{R}_{>0}^{\mathcal{N}}$. We define *Hilbert's projective metric* on $\mathbb{R}_{>0}^{\mathcal{N}}$ by

$$\mathfrak{d}_{\text{H}}(x, y) := \log \frac{\mathfrak{t}(x \odot y^{-1})}{\mathfrak{b}(x \odot y^{-1})}$$

for all x, y in $\mathbb{R}_{>0}^{\mathcal{N}}$ (see [2, 4, 5]). In fact, $\mathfrak{d}_{\text{H}}(\cdot, \cdot)$ is a pseudo-metric on $\mathbb{R}_{>0}^{\mathcal{N}}$. More precisely, $\mathfrak{d}_{\text{H}}(x, y) = 0$ if and only if $x = \beta y$ for some $\beta > 0$. Also, $\mathfrak{d}_{\text{H}}(\alpha x, \beta y) = \mathfrak{d}_{\text{H}}(x, y)$ for all $\alpha, \beta > 0$ and all $x, y \in \mathbb{R}_{>0}^{\mathcal{N}}$. By (5.3) and (6.1), it follows that

$$\begin{aligned} (6.4) \quad \mathfrak{d}_{\text{H}}(x, y) &= \log \mathfrak{t}(x \odot y^{-1}) - \log \mathfrak{b}(x \odot y^{-1}) \\ &= \mathfrak{t}(\text{Log}(x) - \text{Log}(y)) - \mathfrak{b}(\text{Log}(x) - \text{Log}(y)) \\ &= \|\text{Log}(x) - \text{Log}(y)\|_{\text{var}}. \end{aligned}$$

In other words, the mapping Log is an isometry of $(\mathbb{R}_{>0}^{\mathcal{N}}, \mathfrak{d}_{\text{H}})$ into $\ell_{\text{var}}^{\mathcal{N}}$. See [15, 14] for more details. The horofunction boundary of Hilbert's projective metric space $(\mathbb{R}_{>0}^{\mathcal{N}}, \mathfrak{d}_{\text{H}})$ is completely described by combining (6.4) and Theorem 6.2 as follows.

COROLLARY 6.3. *The horofunction boundary of Hilbert's projective metric \mathbf{d}_H on $\mathbb{R}_{>0}^{\mathcal{N}}$ is given by*

$$\partial_H(\mathbb{R}_{>0}^{\mathcal{N}}, \mathbf{d}_H) = \left\{ x \mapsto h_{u,v}(x) \mid \begin{array}{l} u, v \in \mathbb{R}_+^{\mathcal{N}}, \mathbf{t}(u) = 1, \mathbf{t}(v) = 1, \\ u \odot v = 0\mathbb{1} \end{array} \right\},$$

where $h_{u,v}(x) := \log \mathbf{t}(u \odot x) + \log \mathbf{t}(v \odot x^{-1})$ for all $x \in \mathbb{R}_{>0}^{\mathcal{N}}$.

Proof. Let $x \in \mathbb{R}_{>0}^{\mathcal{N}}$. For every $y^n \in \mathbb{R}_{>0}^{\mathcal{N}}$ we have

$$\begin{aligned} \tau_{\mathbf{d}_H}(y^n)(x) &= \mathbf{d}_H(x, y^n) - \mathbf{d}_H(\mathbb{1}, y^n) \\ &= \|\mathrm{Log}(x) - \mathrm{Log}(y^n)\|_{\mathrm{var}} - \|\mathrm{Log}(\mathbb{1}) - \mathrm{Log}(y^n)\|_{\mathrm{var}} \\ &= \tau_{\mathrm{var}}(\mathrm{Log}(y^n))(\mathrm{Log}(x)). \end{aligned}$$

Note that $\mathbf{d}_H(\mathbb{1}, y^n) \rightarrow \infty$ as $n \rightarrow \infty$ if and only if $y^n \rightarrow \xi \in \partial \mathbb{R}_+^{\mathcal{N}}$ as $n \rightarrow \infty$. The latter can be expressed equivalently by $\|\mathrm{Log}(y^n)\|_{\mathrm{var}} \rightarrow \infty$ as $n \rightarrow \infty$. By Theorem 6.2, it follows that $h_\xi \in \partial_H(\mathbb{R}_{>0}^{\mathcal{N}}, \mathbf{d}_H)$ is given by

$$h_\xi(x) = h_{\mu, \nu}^{\mathcal{I}, \mathcal{J}}(\mathrm{Log}(x)) = \mathbf{t}(\mathrm{Log}(x_{\mathcal{I}}) - \mu) + \mathbf{t}(-\mathrm{Log}(x_{\mathcal{J}}) - \nu),$$

where $\emptyset \subsetneq \mathcal{I}, \mathcal{J} \subsetneq \mathcal{N}$ with $\mathcal{I} \cap \mathcal{J} = \emptyset$, and $\mu \in \partial \mathbb{R}_+^{\mathcal{I}}, \nu \in \partial \mathbb{R}_+^{\mathcal{J}}$. Finally, consider $u = (u_i)_{i \in \mathcal{N}}$ and $v = (v_i)_{i \in \mathcal{N}}$ given by

$$u_i = \begin{cases} \exp(-\mu_i), & i \in \mathcal{I}, \\ 0, & i \in \mathcal{N} \setminus \mathcal{I}, \end{cases} \quad v_j = \begin{cases} \exp(-\nu_j), & j \in \mathcal{J}, \\ 0, & j \in \mathcal{N} \setminus \mathcal{J}. \end{cases}$$

Then $u, v \in \mathbb{R}_+^{\mathcal{N}}$ with $\mathbf{t}(u) = 1, \mathbf{t}(v) = 1$ and $u \odot v = 0\mathbb{1}$. Hence, by (5.3) it follows that

$$h_\xi(x) = \log \mathbf{t}(u \odot x) + \log \mathbf{t}(v \odot x^{-1}) = h_{u,v}(x). \quad \blacksquare$$

6.2. Perron's theorem. Let $N \geq 2$ and $\mathcal{N} = \{1, \dots, N\}$. Let $T = (T_{ij}) \in \mathbb{R}^{\mathcal{N} \times \mathcal{N}}$ be a positive matrix, that is, $T_{ij} > 0$ for all $i, j \in \mathcal{N}$. Perron's theorem states that T fixes a unique point in $\mathbb{R}_{>0}^{\mathcal{N}}/\mathbb{R}_{>0}$. We give here a new proof by applying the horofunction boundary of Hilbert's projective metric on $\mathbb{R}_{>0}^{\mathcal{N}}$.

Let $x \in \mathbb{R}_{>0}^{\mathcal{N}}$. For each $i, j \in \mathcal{N}$ we have

$$\begin{aligned} (Tx)_i &:= \sum_{k \in \mathcal{N}} T_{ik} x_k \leq \mathbf{t}((T_{ik})_{k \in \mathcal{N}}) \sum_{k \in \mathcal{N}} x_k, \\ (Tx)_j &:= \sum_{k \in \mathcal{N}} T_{jk} x_k \geq \mathbf{b}((T_{jk})_{k \in \mathcal{N}}) \sum_{k \in \mathcal{N}} x_k. \end{aligned}$$

By Corollary 6.3, each element of the horofunction boundary $\partial_H(\mathbb{R}_{>0}^{\mathcal{N}}, \mathbf{d}_H)$ is of the form $h_{u,v}(x) = \log \mathbf{t}(u \odot x) + \log \mathbf{t}(v \odot x^{-1})$, where $u, v \in \mathbb{R}_+^{\mathcal{N}}$ with $\mathbf{t}(u) = 1, \mathbf{t}(v) = 1$ and $u \odot v = 0\mathbb{1}$. Thus,

$$(6.5) \quad h_{u,v}(Tx) \leq \log \max_{i,j} \left\{ u_i v_j \frac{\mathbf{t}((T_{ik})_{k \in \mathcal{N}})}{\mathbf{b}((T_{jk})_{k \in \mathcal{N}})} \right\}.$$

Let $r_{u,v}$ denote the term on the right-hand side of (6.5). Therefore, for every $x \in \mathbb{R}_{>0}^{\mathcal{N}}$, the sequence $\{Tx, T^2x, T^3x, \dots\}$ stays within the horoball $\mathcal{H}(h_{u,v}, r_{u,v})$. It is well-known [13, 19] that horoballs for Hilbert's projective metric $\mathbf{d}_{\mathbb{H}}$ are convex subsets. It is also well-known [15, 14] that the norm topology and $\mathbf{d}_{\mathbb{H}}$ -topology are equivalent in $\mathbb{R}_{>0}^{\mathcal{N}}/\mathbb{R}_{>0}$. By combining these facts and (6.5), we readily obtain the following.

LEMMA 6.4. *The set*

$$C = \bigcap_{\substack{u,v \in \mathbb{R}_+^{\mathcal{N}} \\ \mathbf{t}(u)=\mathbf{t}(v)=1 \\ u \odot v = 0\mathbf{1}}} \mathcal{H}(h_{u,v}, r_{u,v})$$

is a nonempty convex subset of $\mathbb{R}_{>0}^{\mathcal{N}}$. Furthermore, C is compact in $\mathbb{R}_{>0}^{\mathcal{N}}/\mathbb{R}_{>0}$ and is invariant under the positive matrix T , that is, $TC \subset C$.

We can now consider T as a self-mapping of the compact metric space $(C, \mathbf{d}_{\mathbb{H}})$. In order to prove that T fixes a unique point in $C \subset \mathbb{R}_{>0}^{\mathcal{N}}/\mathbb{R}_{>0}$ we will need the following.

LEMMA 6.5. *Let $x \neq y$ in $\mathbb{R}_{>0}^{\mathcal{N}}/\mathbb{R}_{>0}$. Then $\mathbf{d}_{\mathbb{H}}(Tx, Ty) < \mathbf{d}_{\mathbb{H}}(x, y)$.*

Proof. If $x \neq y$ in $\mathbb{R}_{>0}^{\mathcal{N}}/\mathbb{R}_{>0}$, then there exist $r, s \in \mathcal{N}$ with $r \neq s$ such that

$$\mathbf{b}(x \odot y^{-1}) = \frac{x_r}{y_r} < \frac{x_s}{y_s} = \mathbf{t}(x \odot y^{-1}).$$

On the other hand, for every i ,

$$\begin{aligned} (Tx)_i &= T_{ir}x_r + T_{is}x_s + \sum_{\substack{k \in \mathcal{N} \\ k \neq r,s}} T_{ik}x_k \\ &< T_{ir}y_r \frac{x_s}{y_s} + T_{is} \frac{x_s}{y_s} y_s + \sum_{\substack{k \in \mathcal{N} \\ k \neq r,s}} T_{ik} \frac{x_k}{y_k} y_k \\ &\leq T_{ir}y_r \frac{x_s}{y_s} + T_{is} \frac{x_s}{y_s} y_s + \sum_{\substack{k \in \mathcal{N} \\ k \neq r,s}} T_{ik} \frac{x_s}{y_s} y_k = \frac{x_s}{y_s} (Ty)_i. \end{aligned}$$

The above implies that $\mathbf{t}(Tx \odot (Ty)^{-1}) < \mathbf{t}(x \odot y^{-1})$. In a similar way we can show that $\mathbf{b}(Tx \odot (Ty)^{-1}) > \mathbf{b}(x \odot y^{-1})$. Therefore,

$$\frac{\mathbf{t}(Tx \odot (Ty)^{-1})}{\mathbf{b}(Tx \odot (Ty)^{-1})} < \frac{\mathbf{t}(x \odot y^{-1})}{\mathbf{b}(x \odot y^{-1})},$$

and the result follows. ■

REMARK 6.6. Samelson [17] gives a different proof of Lemma 6.5 by applying projective properties of cross-ratios, which appear in Hilbert's original definition of $\mathbf{d}_{\mathbb{H}}$.

Finally, by combining Lemmas 6.4 and 6.5 and applying Edelstein's fixed-point theorem [7] we obtain the following.

COROLLARY 6.7 (Perron's theorem). *There exists a unique point x^* in $C \subset \mathbb{R}_{>0}^N / \mathbb{R}_{>0}$ such that $T(x^*) = x^*$.*

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